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On some Geometrical and Analytical problems arising from the theory of Isometric Immersion

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The starting point for all results presented in this thesis is the celebrated theory of isometric embeddings of J. Nash and its wide generalization made by M. Gromov in his book “Partial Differential Relations” [1].

I want to thank G. D’Ambra and A. Loi for introducing me to this exciting field and for their help in understanding them as best as possible and making the most out of the found results. I also want to thank T. Gramchev for his help in understanding and studying the analytical points of interest of the theory and A. Kirilov for a pleasant and fruitful collaboration to the analytical part.

The results of Chapter 3 were obtained in collaboration with G. D’Ambra and A. Loi while the results in Chapter 5 were obtained in collaboration with T. Gramchev and A. Kirilov. I am grateful to all of them for a very pleasant collaboration on these works.

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Terminology and Notation

We use the letters M, N and E to denote C^r manifolds. In particular M will always denote a m-dimensional manifold. We denote by $F \xrightarrow{\pi_F} E$ a C^r fibration $\pi_F : F \to E$ while by $G \xrightarrow{\pi_G} E$ we always denote a vector bundle.

The tangent and cotangent bundle of a manifold M are denoted respectively by $TM \xrightarrow{\tau_M} M$ and $T^*M \xrightarrow{\tau^*_M} M$. The total spaces of tensor bundles over M with h contravariant and k covariant indices are denoted by

$$T^h_k M \stackrel{\text{def}}{=} \underbrace{TM \otimes \cdots \otimes TM}_{h \text{ times}} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{k \text{ times}} .$$

The letter $\mathcal{H}$ always denotes a vector subbundle of TM and we use the symbols $\mathcal{H}^h_k$ to denote the tensor products

$$\mathcal{H}^h_k \stackrel{\text{def}}{=} \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{h \text{ times}} \otimes \underbrace{\mathcal{H}^* \otimes \cdots \otimes \mathcal{H}^*}_{k \text{ times}} .$$

Particularly important for us are the vector subbundles $S^0_k M \subset T^0_k M$ and $S\mathcal{H}^0_k \subset \mathcal{H}^0_k$ of totally symmetric tensors.

We denote by $C^r(M,N)$ the space of all C^r maps $M \to N$ and by $\Gamma^r F$ and $\Gamma^r G$ the spaces of all C^r sections of the corresponding fibrations. In particular we use the symbols $\mathfrak{X}(M)$ and $\Omega^1(M)$ for, respectively, $\Gamma^\infty(TM)$ and $\Gamma^\infty(T^*M)$. For every $s \geq r$ the maps

$$i^r_s(M,N) : C^s(M,N) \longrightarrow C^r(M,N) \,,\; i^r_s(F) : \Gamma^s(F) \longrightarrow \Gamma^r(F)$$

denote the canonical inclusions. Finally, we denote by $C^r_+(M)$ the set of all strictly positive C^r functions $M \to \mathbb{R}$.

We endow all these functional spaces with the Whitney strong topology, defined as follows. We recall its definition in the case of $\Gamma^r F$, which is the most general. Let $\Phi = \{(U_i, \psi_i, V_i, \phi_i)\}_{i\in\Lambda}$ a set of locally finite fibered charts of $F \xrightarrow{\pi} E$, i.e. (U_i, ψ_i) is a locally finite cover of E and $(U_i \times V_i, (\psi_i, \phi_i))$ is a locally finite cover of F, $K = \{K_i\}_{i\in\Lambda}$ a set of compact subsets of E s.t. $K_i \subset U_i$ for all $i \in \Lambda$ and $\epsilon = \{\epsilon_i\}_{i\in\Lambda}$ a family of positive numbers. If $f \in \Gamma^r F$ is such that $f(K_i) \subset K_i \times V_i$, then the set

$$\mathcal{U}^r(f, \Psi, K, \epsilon) = \{f' : \|D^k(\psi_i f' \phi_i^{-1}) - D^k(\psi_i f \phi_i^{-1})\| < \epsilon_i\}$$

is a basic set for the strong C^r topology on $F \xrightarrow{\pi} E$. We recall that for compact spaces this reduces to the usual compact-open topology. The reason for our choice is that many subsets of $C^r(M, N)$ important for our thesis are open in this topology. In particular the subset of immersions, embeddings, free maps and, in general, sets defined via an open differential relation are open in $C^r(M, N)$ with the Whitney topology.

We use the following conventions for charts and indices:

1. α, β and γ run from 1 to m;
2. i and j run from 1 to q;
3. a and b run from 1 to q';
4. A and B are used as multindices;
5. a is also used sometimes as index, its range is always declared explicitly.

Manifolds M and E have always dimension m; coordinates on them are denoted by (x^α). The fibers of $F \xrightarrow{\pi_F} E$ have dimension q; fibered coordinates on F are denoted by (x^α, f^i). The fibers of $G \xrightarrow{\pi_G} E$ have dimension q'; fibered coordinates on G are denoted by (x^α, g^a).

We use upper indices for vector (contravariant) components and lower indices for covector (covariant) components; correspondingly, a tensor $t \in \Gamma^r(T^h_k M)$ over a point (x^α) is represented in coordinates by a set of components $(t^{\alpha_1 \dots \alpha_h}_{\beta_1 \dots \beta_k})$. Throughout the thesis we use the Einstein convention of summation over repeated indices, namely the notation $x^\alpha \lambda_\alpha$ always represents the sum $\sum_{\alpha=1}^m x^\alpha \lambda_\alpha$ and similarly for all other indices.

Finally, the following abbreviations are used throughout the paper:

ODE(s)	*Ordinary Differental Equation(s)*
PDO(s)	*Partial Differental Operator(s)*
PDE(s)	*Partial Differental Equation(s)*
rhs	*right hand side*
s_m	$\frac{m(m+1)}{2}$
e_q	*Euclidean metric on* $\mathbb{R}^q$

1 Introduction

The following natural construction is central for the present thesis: once it is given a C^r map $f : M \to N$ between a pair of differential C^r manifolds M and N, its pull-back f^* defines a map[1] from the set of smooth sections $\eta \in \Gamma^r(T^0_k N)$ of all covariant tensor bundles $T^0_k N$ on N to the corresponding sections $f^*\eta \in \Gamma^r(T^0_k M)$ on M. In other words, there exist natural maps

$$\Psi_k : C^r(M,N) \times \Gamma(T^0_k N) \to \Gamma(T^0_k M)$$

defined by $\Psi_k(f,\eta) = f^*\eta$. For every k this map is a PDO of order 1 on the first argument and a linear operator on the second and, after we endow all three functional spaces with the C^r Whitney strong topology, it is continuous in both arguments. The bundle $S^0_2 M$ of symmetric tensors with two covariant (and no contravariant) indices is of great importance in geometry because (pseudo–)Riemannian metrics live in it. Now consider the particular (but fundamental) case $N = \mathbb{R}^q$ and the *isometric operator* PDO

$$\mathcal{D}_{M,q} : C^r(M,\mathbb{R}^q) \to \Gamma(S^0_2 M)$$

defined by $\mathcal{D}_{M,q}(f) = \Psi_2(f, e_q)$, where e_q is the Euclidean metric in $\mathbb{R}^q$. Since we endowed both the source and the target spaces with the Whitney strong topology, the map $\mathcal{D}_{M,q}$ is continuous. Note that the image of the restriction of $\mathcal{D}_{M,q}$ to the (open) subset $\mathrm{Imm}^r(M,\mathbb{R}^q) \subset C^r(M,\mathbb{R}^q)$ of immersions of M into $\mathbb{R}^q$ is contained inside the set of Riemannian metrics on M.

Two natural questions about $\mathcal{D}_{M,q}$ are the following:

1. Is $\mathcal{D}_{M,q}$ surjective?
2. Is $\mathcal{D}_{M,q}$ (or its restriction to some non-empty open set) an open map?

In geometrical language we can reformulate these questions as follows:

[1] Note that this does not hold for the push-forward f_*, which is well-defined only when f is a diffeomorphism.

1. Let g be a Riemannian metric on M. Can we realize g via an immersion of M into $\mathbb{R}^q$?

2. Let f_0 be an immersion of M into $\mathbb{R}^q$ and g_0 the metric induced on M via f_0. If g is a metric C^r-"close enough" to g_0, is there an immersion f, C^r-close to f_0, which induces g on M?

From the analytical point of view, these properties amount to the following:

1. Does the PDE $\mathcal{D}_{M,q}(f) = g$ (see Eq.(2.1) for an expression of this PDE in local coordinates) have smooth solutions for every positive-definite rhs?

2. If $g_0 = \mathcal{D}_{M,q}(f_0)$, does the parametric PDE $\mathcal{D}_{M,q}(f_\lambda) = g_\lambda$, $\lambda \in [0, \epsilon)$, have smooth solutions for small enough λ for every continuous curve g_λ in $\Gamma(S^0_2 M)$?

In both the geometrical and the analytical case, it is also interesting to ask whether the property of being open is true at least for the restriction of $\mathcal{D}_{M,q}$ to some open subset of $C^\infty(M, \mathbb{R}^q)$.

This thesis is dedicated to the study, in different but related contexts, of these two properties for some particular case of isometric operators and other PDOs closely related to them.

1.1 Structure and results of the thesis

The thesis is structured as follows.

In the first three sections of Chapter 2 we review the definitions and main properties of jets, PDOs and free maps in the language used by Gromov in [1]. In Section 2.4 we expose in detail and in our language Gromov's theory of linear undetermined PDOs, which shows that these operator are generically surjective. Finally, in Section 2.5, we move forward towards the proof of a Gromov conjecture by showing that the operators $\mathcal{D}_{\mathbb{R}^m,q}$ are open over a non-empty open set even for $q = n + s_n - 1$, when no free map can arise. The result of this section have been published in [2].

In Chapter 3 we define the concepts of $\mathcal{H}$-immersions and $\mathcal{H}$-free maps, where $\mathcal{H}$ is a subbundle of TM, in such a way that usual immersions and free maps correspond to the subcase $\mathcal{H} = TM$. Then, in Section 3.1, we introduce new PDOs $\mathcal{D}_{\mathcal{H},q}$ and provide conditions under which they are open over a dense open set. Finally, in Section 3.2, we show how to build $\mathcal{H}$-free maps in critical dimension in three geometrically significant types of distributions $\mathcal{H}$. The result of this section have been obtained jointly with G. D'Ambra and A. Loi and have been published in [3].

In Chapter 4 we study the Lie-derivative operators L_ξ when ξ is a vector fields on the plane with no zeros. In particular, in Section 4.1, we prove that, for a generic vector field ξ, the cokernel of L_ξ is infinite-dimensional and, in Section 4.2, that the inequality $L_\xi f > 0$ admits a smooth solution for all vector fields of finite type. Finally, in Section 4.3 we provide a characterization of the set $L_\xi(C^\infty(M))$ and in Section 4.4 we study the behaviour of the functions in $L_\xi(C^\infty(M))$ close to a pair of separatrices. The result of this section are contained in [4].

In Chapter 5 we study the action on functional spaces of a particular case of polynomial Lie-derivative operators on the plane. In Sections 5.1 and 5.2 we study explicitly the behaviour of solutions close to the separatrices. Finally, in Section 5.3 we study in detail the action of the inverses of these operators. The result of this section have been obtained jointly with T. Gramchev and A. Kirilov and are contained in [5].

In the following two sections we briefly illustrate the most relevant results which are at the base of the present thesis.

1.2 The Nash embedding theorem and the Newton-Nash-Moser-Gromov IFT

The isometric embedding problem is the natural generalization, to the field of Riemannian Geometry, of the classic Whitney embedding theorem:

Theorem 1.2.1 (Whitney, 1944)**.** *Every m-dimensional manifold M admits an embedding into $\mathbb{R}^{2m}$ and an immersion into $\mathbb{R}^{2m-1}$.*

A fundamental consequence of Whitney's theorem is that the concept of (real) smooth manifold is not more general than the concept of submanifold of the euclidean space. This fact is not trivial since a similar statement would be false, for example, in the complex case, where no compact manifold can be biholomorphically embedded into any $\mathbb{C}^q$ by Liouville's theorem. It is hence natural asking whether also every Riemannian manifold is a Riemannian submanifold of some euclidean space.

The first publication about this topic goes back to 1873, when Schlaefli [6] conjectured that, for the existence of a local isometrical immersion $f : M \to \mathbb{R}^q$, it is enough that $q \geq s_n$, where $s_n = n(n+1)/2$ is the number of unknowns of the equation $\mathcal{D}(f) = g$ (see Section 2.3 for an expression of this PDE in coordinates). This conjecture was proved in 1926 for C^ω-immersions by Janet [7] in the 2-dimensional case and then, a year later, in the general case by Cartan [8] as an application of his theory of exterior differential systems. It was a striking and unexpected discovery, made by Nash [9] in 1954 and refined the next year by Kuiper [10], that the properties of isometric immersion, even locally, depend strongly on their regularity. The surprise that these results caused in the scientific community is well expressed by the following sentence of Gromov extracted from a recent interview [11]:

At first, I looked at one of Nash's papers and thought it was just nonsense. But Professor Rokhlin said: No, no. You must read it. I still thought it was nonsense; it could not be true. But then I read it, and it was incredible. It could not be true but it was true. There were three papers; the two more difficult ones, *on embeddings, they looked nonsensical. Then you look at the way it is done, and you also think that it looks nonsensical. After understanding the idea you try to do it better; many people tried to do it in a better way. But when you look at how they were doing it, and also what I tried, and then come back to Nash, you have to admit that he had done it in a better way. He had a tremendous analytic power combined with geometric intuition. This was a fantastic discovery for me: how the world may be different from what you think!*

From the global (see below) result of Nash it can be extracted the following local corollary (see [12], Section 1.2.6):

Theorem 1.2.2 (Nash, 1954; Kuiper, 1955)**.** *Let M be an m-dimensional C^1-manifold. Then every point of M has a neighbourhood which admits an isometric C^1-immersion into $\mathbb{R}^{m+1}$.*

Clearly the regularity cannot be improved since the curvature of the metric is an invariant for C^2 maps. Indeed, two years later, Nash [13] found a much higher upper bound for the dimension of the target space in case of more regular immersions, namely:

Theorem 1.2.3 (Nash, 1956)**.** *Let M be a m-dimensional C^r-manifold, $r \geq 3$. Then every point of M has a neighbourhood which admits an isometric C^r-immersion into $\mathbb{R}^{4m+s_m}$.*

In this case the dimension of the target space is not sharp and was improved by Gromov (see below) to $m^2 + 10m + 3$ for $r = 3$ and to $(m+2)(m+3)/2$ for $r \geq 4$. Note that the case $r = 2$, not covered by the two theorems above, is still an open problem.

Obstructions to the global extension of local isometric immersions come sometimes from the topology and sometimes from the (Riemannian) geometry. An example of the first case comes from the fact that, clearly, no compact m-dimensional manifold M can be immersed into $\mathbb{R}^m$ and so *a fortiori* no isometric immersion $M \to \mathbb{R}^m$ can exist. The first example of the second case goes back to Hilbert [14], that showed that the the Lobachevskii plane (i.e. the plane endowed with a metric of constant negative curvature) cannot be isometrically C^2-embedded into $\mathbb{R}^3$; this result was much later generalized by Efimov [15], which proved that no metric on the plane with curvature bounded above by a negative number can be induced via a C^2-immersion into $\mathbb{R}^3$. An example of Gromov (see [12], Appendix 3) shows that the disk admits a C^2-open set of metrics which cannot be C^2-immersed into $\mathbb{R}^3$, proving that such behaviour are not limited to open manifolds. Obstructions might also come from both sources: for example the elliptic plane (i.e. the projective plane with the canonical metric) does not admit any C^2-immersion into $\mathbb{R}^3$ for it is non-orientable while every surface of positive curvature immersed in $\mathbb{R}^3$ must be orientable.

Before Nash, the only significative result on the global existence of isometric immersions was due to a problem posed by Weyl [16] about whether every analytic positive-curvature metric on the sphere comes from some analytic immersion of $\mathbb{S}^2$ into $\mathbb{R}^3$. The problem was attacked and solved by Alexandrov and Pogorelov from a geometric point of view and by Lewy and Niremberg from an analytic point of view (see Refs. in [13]). It is then easily imaginable the enormous impact on this field of the following global results of Nash:

Theorem 1.2.4 (Nash, 1954; Kuiper, 1955)**.** *Let M be an m-dimensional C^1-manifold. If M admits a strictly short*[2] *immersion (resp. embedding) into $\mathbb{R}^q$ and $q \geq m+1$ then it also admits a C^1 isometric immersion (resp. embedding) into $\mathbb{R}^q$. Moreveor, strictly short immersions (resp. embeddings) $M \to \mathbb{R}^q$ always exist for $q \geq 2m$ (resp. $q \geq 2m+1$).*

This theorem was first proved by Nash for $q \geq m+2$ and, a year later, extended to the case $q \geq m+1$ by Kuiper. As a corollary, we get that every m-dimensional manifold can be C^1 isometrically embedded into $\mathbb{R}^{2m+1}$ and C^1 immersed into $\mathbb{R}^{2m}$. Moreover in the compact case the dimensions of the target space can be reduces by 1 for both the embeddings and the immersions. For example, this implies the astonishing fact that every surface can be C^1 isometrically immersed into $\mathbb{R}^3$.

Two years later Nash proved the following theorem about more regular immersions:

Theorem 1.2.5 (Nash, 1956)**.** *Every m-dimensional closed (resp. open) Riemannian C^r-manifold, $r \geq 3$, admits an isometric C^r-immersion into $\mathbb{R}^q$ for $q = 4m + 3s_m$ (resp. $q = (4m+3s_m)(m+1)$).*

These bounds were then improved by Gromov to $q = m^2 + 10m + 3$ (e.g. see [1], Sec. 3.1.1). In terms of the isometric operator, we can restate these results of Nash in the following way:

Theorem 1.2.6 (Nash,Kuiper,Gromov)**.** *If M is a C^1-manifold, the isometric operator $\mathcal{D}_{M,q} : C^1(M, \mathbb{R}^q) \to \Gamma(S^0_2 M)$ is surjective for $q \geq 2m$. If M is a C^r-manifold, with $r \geq 3$, then $\mathcal{D}_{M,q} : C^r(M, \mathbb{R}^q) \to \Gamma(S^0_2 M)$ is surjective for $q \geq m^2 + 10m + 3$.*

The relevance of these results for the present thesis lies even more in their proof than in their content. Indeed, in order to prove them, Nash introduced a clever infinite-dimensional implicit function theorem, improved later by Moser and other authors and that was used since then in several contexts related to PDEs, including the celebrated KAM theorem.

In his book "Partial Differential Relations" [1], Gromov widely generalized the method of Nash to any PDO. For the purposes of the present thesis, the version of this Newton-Nash-Moser-Gromov Implicit Function Theorem can be stated (see also Section 2.2) as follows:

Theorem 1.2.7 (Newton-Nash-Moser-Gromov IFT)**.** *Let $\mathcal{D}$ be a smooth PDO which is infinitesimally invertible over some open subset U. Then the restriction of $\mathcal{D}$ to U is an open map.*

[2] A *strictly short* immersion (resp. embedding) $f : (M, g) \to (\mathbb{R}^q, e_q)$ is an immersion (resp. embedding) s.t. $g - f^* e_q$ is a metric on M.

Loosely speaking, an operator $\mathcal{D}$ is infinitesimally invertible if its linearization is invertible (for a formal definition of infinitesimal invertibility see Section 2.2).

In case of isometries, free maps play an important role. We recall that a map $f : M \to \mathbb{R}^q$ is free if the vectors of its first and second partial derivatives are all linearly independent (see Section 2.3). It is easy to verify that the isometric operator $\mathcal{D}_{M,q}$ is infinitesimally invertible over the set of free maps (see Section 2.3). Hence Newton-Nash-Moser-Gromov IFT leads immediately to the following famous theorem of Nash:

Theorem 1.2.8 (Nash 1956)**.** *Let M be a C^r manifold, with $r \geq 3$. The isometric operator $\mathcal{D}_{M,q}$ is open over the set of free maps from M to $\mathbb{R}^q$.*

1.3 The Cohomological Equation for regular vector fields in $\mathbb{R}^2$

In Chapter 3 we show that, given a C^1 map $f : M \to \mathbb{R}$ and a vector field $\xi \in \mathfrak{X}(M)$ without zeros, on the foliation $\mathcal{F}_\xi$ of integral trajectories of ξ it is induced a symmetric tensor with two covariant indices $\mathcal{D}_\xi(f) = (L_\xi f)^2\theta^2$ for some 1-form $\theta \in \Omega^1(M)$ such that $\theta(\xi) = 1$. This quadratic form is a metric iff $L_\xi f > 0$, so $\mathcal{D}_\xi$ is surjective iff L_ξ is surjective on the subspace of positive functions. Similarly, if $g_0 = \mathcal{D}_\xi(f_0)$ is a metric on $\mathcal{F}_\xi$ and $g_\epsilon = g_0 + \epsilon\delta g$ is a small perturbation of g_0 then the linearized version of $g_\epsilon = \mathcal{D}_\xi(f_\epsilon)$ is $L_\xi \delta f = \delta g/2$, so $\mathcal{D}_\xi$ is an open map close to f_0 iff L_ξ is surjective.

More generally, the problem of the solvability of the so-called cohomological equation

$$L_\xi f = g \tag{1.1}$$

in dependence of the topology of the foliation of its integral trajectories is relevant in the context of dynamical systems and was recently studied from two complementary points of view:

1. The time-change in the flow induced by the multiplication of ξ by a strictly positive smooth function λ is *trivial* (i.e. ξ and $\lambda\xi$ belong to the same smooth conjugacy class) iff $\lambda - 1 \in L_\xi(C^\infty(M))$ (see [17, 18] for more details).

2. In a series of papers (see [19] and the works cited therein) S.P. Novikov introduced exotic cohomological theories related to dynamical systems on manifolds and showed that some cohomology groups associated to the cohomological equation are related with the equivariant homology obtained by considering the set of the invariant differential forms considered as forms on the leaves space.

Note that the question of the solvability of the cohomological equation is of purely global nature: it is well known indeed that, for every point $p \in M$ with $\xi_p \neq 0$, there is a neighbourhood U_p s.t. $L_\xi(C^\infty(U_p)) = C^\infty(U_p)$, so that the cohomological equation is always solvable. All solutions to (1.1) are given explicitly by

$$f(p) = F(p_l) + \int_0^{\lambda_p} \left[\phi_\xi^\lambda(p_l)\right]^* g \, d\lambda$$

where ϕ_ξ^λ is the flow of ξ, λ_p the time to reach p from the point p_l lying on a fixed local transversal line l and F any smooth function defined on l.

The most general result known on this subject is perhaps the following theorem [20]:

Theorem 1.3.1 (Hormander and Duistermaat, 1972). *Let M^m be an open connected manifold and ξ a vector field without zeros on it. Then the following are equivalent:*

1. $L_\xi(C^\infty(M)) = C^\infty(M)$;
2. $(L_\xi + a)(C^\infty(M)) = C^\infty(M)$ *for any* $a \in C^\infty(M)$;
3. ξ *admits a global transversal, i.e. a codimension-1 embedded surface* $N \subset M$ *which is transversal to* ξ *at every point and cuts every of its integral curves exactly once.*

It is a classical observation, going back to Siegel [21] and related to the small divisors problems that led ultimately to the KAM theory (e.g. see [22] and [23]), that the Lie derivative operators can be easily non-surjective. Consider indeed the vector field $\xi = \partial_x + \alpha\partial_y$ on $\mathbb{T}^2$. It is well-known that, for a generic $\alpha \in \mathbb{R}$, the only solvability condition for $L_\xi f = g$ is the obvious $\int_{\mathbb{T}^2} g \, d\mu = 0$, where $d\mu$ is the Haar measure on $\mathbb{T}^2$. Indeed, by developing f and g in Fourier series, the cohomological equation writes

$$2\pi i(m + \alpha n) f_{m,n} = g_{m,n} \, .$$

The corresponding solution

$$f(x, y) = \frac{1}{2\pi i} \sum_{(m,n)\neq(0,0)} \frac{g_{m,n}}{m + \alpha n} \exp\{2\pi i(mx + ny)\}$$

does converge to a C^∞ function on $\mathbb{T}^2$ if α is generic, so that $m + \alpha n$ does not grow too fast. Then in this case $\dim \operatorname{coker} L_\xi = 1$. If instead α is a Liouville number, i.e. a number such that for every $n \in \mathbb{N}$ there exist two integers p and q such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n} \, ,$$

then the denominators in the Fourier coefficients of f grow too fast and we have $\dim \operatorname{coker} L_\xi = \infty$.

Recently there was a renewed interest in the cohomological equation on (compact) surfaces. In Nineties Forni [18] generalized this classical results to orientable surfaces M of any genus for the action of the Lie derivatives L_ξ acting as a weak derivative on the Sobolev spaces $H^k(M)$, namely the spaces of L^2 functions on M whose weak derivatives, up to order k, also belong to $L^2(M)$. In particular he proved the following results:

Theorem 1.3.2 (Forni, 1995). *Let ω be a symplectic form on a compact surface M of genus $g \geq 2$. Then for a generic Hamiltonian vector field ξ with set of zeros Σ there exists a $k > 0$ s.t. if g is compactly supported in $M \setminus \Sigma$ and $\int_M g\,\omega = 0$ then the cohomological equation $L_\xi f = g$ on $H^k(M)$ admits has a solution $f \in L^2_{loc}(M \setminus \Sigma)$.*

Theorem 1.3.3 (Forni, 1995). *Let ω be a symplectic form on a compact surface M of genus $g \geq 2$. Then for a generic Hamiltonian vector field ξ with set of zeros Σ and for any $s > 2g-2$ there is a finite number of distributions $d_1, \dots d_{n_s} \in H^{-s}_{loc}(M \setminus \Sigma)$ such that the cohomological equation $L_\xi f = g$ on $H^s(M)$ has a solution $f \in H^{s-2g-2}(M)$ if g satisfies the following compatibility conditions:*

$$\int_M g\,\omega = 0\,,\ d_1(g) = 0\,,\ \dots\,,\ d_{n_s}(g) = 0$$

In 2007 Novikov [19], in a general work where he introduced exotic cohomology groups associated to Hamiltonian dynamical systems, showed that in the smooth setting the situation is rather similar to the case of Liouvillian constant vector fields in the 2-torus:

Theorem 1.3.4 (Novikov, 2007). *Let ξ be a generic Hamiltonian vector field on a symplectic compact surface M of genus $g \geq 2$ and $L_\xi : C^\infty(M) \to C^\infty(M)$ the corresponding Lie derivative operator. Then* $\dim \operatorname{coker} L_\xi = \infty$.

2

Free Maps and Infinitesimal Invertibility of the Isometric Operator

The proof of the celebrated theorem of Nash of isometric embeddings of a manifold M into the Euclidean space $\mathbb{R}^q$ (see Theorem 1.2.6) is proved in two steps. The first step is algebraic and consists in the construction of an inverse to the linearization of the isometric operator $\mathcal{D}_{M,q}$, leading to the definition of free maps. The second step, much harder, is analytic and consists in an infinite-dimensional Implicit Function Theorem (IFT) (see Theorem 2.2.8) that shows how it is possible to overcome the loss of derivatives in going from solutions of linearized problem to solutions of the original problem.

In this chapter, following the point of view of Gromov in [1], we first introduce our notations for Jet Spaces in Section 2.1. Then, in Section 2.2, we define PDOs and the notion of infinitesimal invertibility in order to state the Newton-Nash-Moser-Gromov IFT. Next we define Free maps and illustrate the Nash embedding theorem for C^r maps, $r \geq 3$ (Section 2.3). All definitions and theorems presented in the two previous sections are extracted from [1], Section 2.3.2.

In Section 2.4 we illustrate in detail Gromov's theory of linear under-determined PDOs, leading ultimately to Theorem 2.4.17, claiming that in the generic case such PDOs are surjective. Finally, in Section 2.5 we answer positively, in a particular case, to a question posed by Gromov in [1] using the Theorem of Duistermaat and Hormander 1.3.1 (these results have been published in [2]) and then we make some step towards the general case using arguments similar to those illustrated in the previous section.

2.1 Jet Spaces

Jet spaces of order $k = 0, 1, 2, \ldots,$ are a finite-dimensional approximation of the functional spaces of C^∞ functions and constitute the natural geometrical setting for PDEs in the same way the tangent bundles constitute the natural geometrical setting for ODEs.

Consider first the set $C^\infty(\mathbb{R}^m, \mathbb{R}^q)$ and denote by t^k the operator which associates to any function $\psi \in C^\infty(\mathbb{R}^m, \mathbb{R}^q)$ its value and the value of all of its derivatives up to order k at 0, i.e.

$$t^k(\psi) = \big(\psi(0), D\psi(0), \dots, D^k\psi(0)\big) ,$$

where

$$D\psi(0) : T_0\mathbb{R}^m \to T_{\psi(0)}\mathbb{R}^q$$

is represented in coordinates by the Jacobian matrix $(\partial_\alpha \psi^i(0))$ of ψ at 0 and, more generally, the multilinear maps

$$D^\ell\psi(0) : (T_0\mathbb{R}^m)^\ell \to T_{\psi(0)}\mathbb{R}^q$$

are represented by the matrices of all derivatives of order ℓ of the components of ψ evaluated at 0, i.e.

$$D^\ell\psi(0) = (\partial_{\alpha_1 \dots \alpha_l}\psi^i(0)) .$$

We call $t^k(\psi)$ the *jet* of ψ of order k at 0. The relation $\sim_k$ defined by $\psi_1 \sim_k \psi_2$ iff $t^k(\psi_1) = t^k(\psi_2)$ is clearly an equivalence relation and the quotient

$$J^k_0(\mathbb{R}^m, \mathbb{R}^q) \stackrel{\text{def}}{=} C^\infty(\mathbb{R}^m, \mathbb{R}^q)/ \sim_k$$

is the space of all such jets. Any coordinates system (y^i) on $\mathbb{R}^q$ induces coordinates $(y^i, y^i_\alpha, \dots, y^i_{\alpha_1 \dots \alpha_k})$ on $J^k_0(\mathbb{R}^m, \mathbb{R}^q)$ such that

$$y^i(\psi) = \psi(0)\,,\ y^i_\alpha(\psi) = \partial_\alpha\psi(0)\,,\ \dots\,,\ y^i_{\alpha_1 \dots \alpha_k}(\psi) = \partial_{\alpha_1 \dots \alpha_k}\psi(0)$$

Clearly, since derivatives commute with each other, part of these coordinates are actually redundant; in order to have a true coordinate system we keep only those such that the lower indices $\alpha_1 \dots \alpha_h$ are in non-decreasing order, i.e. $\alpha_1 \le \dots \le \alpha_h$ for all $h = 1, \dots, k$. In general we have that

$$J^k_0(\mathbb{R}^m, \mathbb{R}^q) \simeq \mathbb{R}^q \oplus \left[\mathbb{R}^q \otimes \left(\oplus^k_{\ell=1} S^0_\ell\, \mathbb{R}^m\right)\right] ,$$

where the $S^0_\ell\,\mathbb{R}^m$ are the vector bundles of the symmetric tensor products of ℓ copies of $T^*\mathbb{R}^m$. In particular

$$\dim J^k_0(\mathbb{R}^m, \mathbb{R}^q) = q\left(1 + \binom{m}{1} + \binom{m+1}{2} + \dots + \binom{m+k-1}{k}\right) = q\binom{m+k}{k} ,$$

where $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ is the binomial coefficient. For every $h \le k$ we have natural projections

$$\pi^h_k : J^k_0(\mathbb{R}^m, \mathbb{R}^q) \to J^h_0(\mathbb{R}^m, \mathbb{R}^q)$$

which define in general *affine* bundles. There are two kinds of projections particularly important. The first is

$$\pi^k_{k+1} : J^{k+1}_0(\mathbb{R}^m, \mathbb{R}^q) \to J^k_0(\mathbb{R}^m, \mathbb{R}^q) ,$$

which defines, for every k, an *affine* bundle such that

$$\dim_{J_0^k(\mathbb{R}^m,\mathbb{R}^q)} J_0^{k+1}(\mathbb{R}^m,\mathbb{R}^q) = q\binom{m+k}{k+1},$$

where by $\dim_E F$ we denote the dimension of the fiber of a fibration of F over E (it will be clear from the context to which fibration we refer to).

The second is

$$\pi_k^0 : J_0^k(\mathbb{R}^m,\mathbb{R}^q) \to \mathbb{R}^q\,,$$

whose fiber $\pi_k^0(y)$ is the set of the k-jets of all functions s.t. $\psi(0) = y$ and is denoted by $J_0^k(\mathbb{R}^m,\mathbb{R}^q)_y$.

The space of *all* jets from $\mathbb{R}^m$ to $\mathbb{R}^q$ is defined as

$$J^k(\mathbb{R}^m,\mathbb{R}^q) \stackrel{\text{def}}{=} \bigcup_{x\in\mathbb{R}^m, y\in\mathbb{R}^q} J_x^k(\mathbb{R}^m,\mathbb{R}^q)_y \simeq \mathbb{R}^m \oplus \mathbb{R}^q \oplus \left[\mathbb{R}^q \otimes \left(\oplus_{\ell=1}^k S_\ell^0\,\mathbb{R}^q\right)\right]$$

To every map $f \in C^k(\mathbb{R}^m,\mathbb{R}^q)$ it is naturally induced a map $j^k(f)$, called *prolongation* of f, defined as

$$\begin{aligned} j^k(f) : \mathbb{R}^m &\to J^k(\mathbb{R}^m,\mathbb{R}^q) \\ (x^\alpha) &\mapsto (x^\alpha, f^i(x^\alpha), \partial_{\alpha_1} f^i(x^\alpha), \ldots, \partial_{\alpha_1\ldots\alpha_k} f^i(x^\alpha)) \end{aligned}$$

Once a map $F \in C^k(\mathbb{R}^q,\mathbb{R}^p)$ is given, we can send jets $j^k(f) : \mathbb{R}^m \to J^k(\mathbb{R}^m,\mathbb{R}^q)$ to new jets

$$j^k(F\circ f) : \mathbb{R}^m \to J^k(\mathbb{R}^m,\mathbb{R}^p)\,;$$

similarly, given a $G \in C^k(\mathbb{R}^m,\mathbb{R}^n)$ we can move jets $j^k(f) : \mathbb{R}^n \to J^k(\mathbb{R}^n,\mathbb{R}^q)$ to

$$j^k(f\circ G) : \mathbb{R}^m \to J^k(\mathbb{R}^m,\mathbb{R}^q)\,.$$

This induces jet bundle morphisms

$$J^k(\mathbb{R}^m, F) : J^k(\mathbb{R}^m,\mathbb{R}^q) \to J^k(\mathbb{R}^m,\mathbb{R}^p)\,,$$

$$J^k(G, \mathbb{R}^q) : J^k(\mathbb{R}^n,\mathbb{R}^q) \to J^k(\mathbb{R}^m,\mathbb{R}^q)\,.$$

Example 2.1.1. *The space of 1-jets* at *0 of applications* $\mathbb{R} \to \mathbb{R}^q$ *is exactly the tangent bundle of* $\mathbb{R}^q$*:*

$$J_0^1(\mathbb{R},\mathbb{R}^q) \simeq T\mathbb{R}^q\,.$$

Under this identification, the map $J_0^1(\mathbb{R},F) : J_0^1(\mathbb{R},\mathbb{R}^q) \to J_0^1(\mathbb{R},\mathbb{R}^p)$ *associated to any* C^1 *map* $F : \mathbb{R}^q \to \mathbb{R}^p$ *coincides with the tangent map* $TF : T\mathbb{R}^q \to T\mathbb{R}^p$*.*

Similarly, the space of 1-jets to *0 of applications* $\mathbb{R}^m \to \mathbb{R}$ *is exactly the cotangent bundle of* $\mathbb{R}^m$*:*

$$J^1(\mathbb{R}^m,\mathbb{R})_0 = T^*\mathbb{R}^m\,.$$

The map $J^1(F,\mathbb{R})_0 : J^1(\mathbb{R}^m,\mathbb{R})_0 \to J^1(\mathbb{R}^n,\mathbb{R})_0$ *coincides with the cotangent map* T^*F*.*

A very important case is given by the projection of the trivial bundle

$$\pi_1 : \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^m \,,$$

so that

$$J^k(\mathbb{R}^m, \pi_1) : J^k(\mathbb{R}^m, \mathbb{R}^m \times \mathbb{R}^q) \to J^k(\mathbb{R}^m, \mathbb{R}^m) \,.$$

We denote by $\Gamma^r(\pi_1)$ the set of C^r *sections* of this bundle, i.e. of those C^r maps $f : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^q$ such that $\pi_1 \circ f = \mathrm{id}_{\mathbb{R}^m}$. Then, in coordinates,

$$j^k(f)(x^\alpha) = (x^\alpha, x^\alpha, y^i, \delta^\alpha_\beta, \partial_\beta y^i, 0, \partial_{\beta_1\beta_2} y^i, \dots, 0, \partial_{\beta_1\dots\beta_k} y^i) \,,$$

so that

$$J^k(\mathbb{R}^m, \pi_1)(j^k(f)) = j^k(\pi_1 \circ f) = j^k(\mathrm{id}_{\mathbb{R}^m}) = (x^\alpha, x^\alpha, \delta^\alpha_\beta, 0, \dots, 0)$$

and, viceversa, every map whose jet is sent into $j^k(\mathrm{id}_{\mathbb{R}^m})$ by $J^k(\mathbb{R}^m, \pi_1)$ is a section of $\mathbb{R}^m \times \mathbb{R}^q \xrightarrow{\pi_1} \mathbb{R}^m$. We denote by $J^k(\pi_1)$ the (closed) submanifold

$$J^k(\pi_1) \stackrel{\text{def}}{=} J^k(\mathbb{R}^p, \pi_1)^{-1}(j^k(\mathrm{id}_{\mathbb{R}^p})) \subset J^k(\mathbb{R}^p, \mathbb{R}^p \times \mathbb{R}^m) \,,$$

which contains the k-jets of all sections of the bundle.

Now consider two C^k-manifolds M and N. Since each of them is locally euclidean and the construction of jet spaces is natural, the spaces of jets $J^k_x(M,N)_y$, with $x \in M$ and $y \in N$, built via any pair of coordinate systems do not depend on the arbitrary choice of them and so the spaces

$$J^k(M,N) \stackrel{\text{def}}{=} \bigcup_{x \in M, y \in N} J^k_x(M,N)_y$$

are intrinsically well defined and similarly are well defined the projections

$$\pi^h_k : J^k(M,N) \to J^h(M,N)$$

for every $h \leq k$. In particular all the $J^k(M,N)$ fiber naturally over $J^0(M,N) \simeq M \times N$.

Example 2.1.2. *The isomorphisms in Example 2.1.1 still hold after replacing $\mathbb{R}^m$ with a manifold M and $\mathbb{R}^q$ with a manifold N. Other noteworthy particular cases are the space of 1-jets of maps $M \to N$,*

$$J^1(M,N) \simeq T^*M \otimes TN \,,$$

whose sections are the linear homomorphisms between TM and TN, and the space of 1-jets of diffeomorphisms at 0 *between $\mathbb{R}^m$ and a manifold M of same dimension, $G^1_0(\mathbb{R}^m, M)$, which is isomorphic to the principal bundle $L(M)$ of the m-frames over M.*

Finally, let $F \xrightarrow{\pi} E$ a C^∞ fibration of a manifold F, with $\dim F = m+q$, over a manifold E, with $\dim E = m$. Then each fiber of the fibration is q-dimensional and, similarly to what we did above in case of the trivial bundle, we can define the bundle of k-jets of sections of $F \xrightarrow{\pi} E$ as the set

$$J^kF \stackrel{\text{def}}{=} J^k(E,\pi)^{-1}\left(j^k(\mathrm{id}_E)\right),$$

with

$$\dim J^kF = q\binom{m+k}{k}, \quad \dim_{J^kF} J^{k+1}F = q\binom{m+k}{k+1}.$$

Occasionally, depending on the opportunity, the bundle J^kF will be also denoted by $J^k\pi$. Correspondingly, the spaces of C^r sections of $F \xrightarrow{\pi} E$ will be denoted sometimes by Γ^rF and sometimes by $\Gamma^r\pi$.

The jet space of sections of a bundle is general enough to include also the spaces of jets between two manifolds M and N. Indeed every map $f : M \to N$ can be seen a section $\tilde{f}$ of the trivial bundle $\pi_M : M \times N \to M$ defined by $\tilde{f}(x) = (x, f(x))$, so that

$$J^k\pi_M \simeq J^k(M,N), \quad \Gamma^r\pi_M \simeq C^r(M,N).$$

From this moment on then we will follow Gromov's approach and consider just the case of jets of sections of a fibration.

2.2 PDOs, infinitesimal invertibility and the Newton-Nash-Moser-Gromov theorem

Let $F \xrightarrow{\pi_F} E$ be a C^∞-fibration and $G \xrightarrow{\pi_G} E$ a vector bundle over the same manifold E.

Definition 2.2.1. *A C^k PDO over F of order r with values in G is a map*

$$\mathcal{L}_r : \Gamma^rF \to \Gamma^0G$$

whose coefficients, written in any coordinate system, are all of class C^k and whose value on a section $f \in \Gamma^rF$ at a point $x \in E$ depends only on j^r_xf.

Denote by (x^α) the coordinates on E and by (x^α, f^i) and (x^α, g^a) fibered coordinates respectively on F and G. Then the induced coordinates on J^rF are $(x^\alpha, f^i, f^i_\alpha, \ldots, f^i_{\alpha_1\ldots\alpha_r})$ and $\mathcal{L}_r$ writes as

$$\mathcal{L}_r(f)(x^\alpha) = (\Lambda^a_r(x^\alpha, f^i(x^\alpha), \partial_\alpha f^i(x^\alpha), \ldots, \partial_{\alpha_1\ldots\alpha_r}f^i(x^\alpha)).$$

where $\Lambda_r = (\Lambda^a_r) : J^rF \to G$ is some C^k map.

The equation

$$\mathcal{L}_r(f) = \phi$$

then is represented by the PDE system

$$\Lambda^a_r(x^\alpha, f^i(x^\alpha), \partial f^i_\alpha(x^\alpha), \dots, \partial f^i_{\alpha_1\dots\alpha_r}(x^\alpha)) = \phi^a$$

of q' equations (since $a = 1, \dots, q'$) in q unknowns (the functions $f^1, \dots, f^q$). This suggests a new equivalent definition for PDOs over F:

Definition 2.2.2. *A C^k PDO over F of order r is a C^k map*

$$\Lambda_r : J^r F \to G$$

Observe that, when the fibration $F \xrightarrow{\pi_F} E$ is trivial (namely when $F = E \times Q$ for some manifold Q and π_F is the projection on the first factor), the space of C^r sections of F is naturally isomorphic to the space of C^r functions from E to Q, i.e. $\Gamma^r(F) \simeq C^r(E,Q)$, and, equivalently, $J^r F \simeq J^r(E,Q)$.

Example 2.2.3. *The simplest C^k PDO over a manifold M is represented by a vector field $\xi \in \Gamma^s(\tau_M)$, i.e. a C^k section of the tangent bundle $TM \xrightarrow{\tau_M} M$. We denote the corresponding first-order linear homogeneous PDO by L_ξ (Lie derivative in the ξ direction). L_ξ is a map from $C^1(M)$ to $C(M)$, i.e. here $E = M$ and $F = G = M \times \mathbb{R}$. In coordinates $L_\xi = \xi^\alpha \partial_\alpha$ and the corresponding map $\Lambda_\xi : J^1(M, \mathbb{R}) \to M \times \mathbb{R}$ is defined as*

$$\Lambda_\xi(x^\beta, f, f_\beta) = \xi^\alpha(x^\beta) f_\alpha \,.$$

The corresponding PDE

$$(j^1 f)^* \Lambda_\xi = \phi$$

is called cohomological equation *and will be studied in detail in Chapter 3 and Chapter 4 for $M = \mathbb{R}^2$ in case of vector fields with no zeros.*

Example 2.2.4. *The most important PDO for this thesis is the* isometric operator*, namely the C^∞ quadratic first-order operator defined by $\mathcal{D}_{M,q}(f) = f^* e_q$, where $f \in C^1(M, \mathbb{R}^q)$ and e_q is the euclidean metric on $\mathbb{R}^q$. Here $E = M$, $F = M \times \mathbb{R}^q$ and $G = S^0_2 M$ (since $\mathcal{D}_{M,q}(f)$ is a symmetric tensor with two contravariant indices), so that*

$$\mathcal{D}_{M,q} : C^1(M, \mathbb{R}^q) \to \Gamma^0(S^0_2 M) \,.$$

In coordinates

$$\mathcal{D}_{M,q}(f) = \delta_{ij} \partial_\alpha f^i \partial_\beta f^j \,,$$

so that the corresponding map $\Lambda_{M,q} : J^1(M, \mathbb{R}^q) \to S^0_2 M$ is defined as

$$\Lambda_{M,q}(x^\alpha, f, f_\alpha) = \delta_{ij} f^i_\alpha f^j_\beta \,.$$

Now recall that a vector $v \in T_p F$, $p \in F$, of the fibered manifold $F \xrightarrow{\pi_F} E$ is called *vertical* if $T_p \pi_F(v) = 0$ and consider the vector bundle of all vertical vectors $VF \stackrel{\text{def}}{=} \ker T\pi_F \subset TF$. For every section $f \in \Gamma^r F$ we can build the pull-back bundle $f^*(VF) \stackrel{f^*\pi_F}{\longrightarrow} E$, namely the vector bundle over E having over every point $x \in E$ the vector space $V_{f(x)}F$.

The space $\Gamma^r_f \stackrel{\text{def}}{=} \Gamma^r(f^*(VF))$ of the C^r sections of this new bundle can be thought as the tangent space at f of the (infinite dimensional) space $\Gamma^r F$. Indeed consider a 1-parameter family f_t of sections which is C^1 with respect to the parameter t and such that $f_0 = f$. For any $x_0 \in E$ we get a curve $f_t(x_0) \subset F$ whose tangent vector $\eta_f(x_0) = df_t(x_0)/dt|_{t=0}$ lies by construction over the point $f(x_0)$ and is clearly vertical since

$$T_{x_0}\pi_F(\eta_f(x_0)) = T_{x_0}\pi_F\left(\left.\frac{df_t(x_0)}{dt}\right|_{t=0}\right) = \left.\frac{d(\pi_F \circ f_t)(x_0)}{dt}\right|_{t=0} = \left.\frac{dx_0}{dt}\right|_{t=0} = 0\,.$$

Hence the way the section $\eta_f \in \Gamma^r_f$ is defined out of f is completely analogous to the one used to define tangent vectors to a manifold over some point in the finite-dimensional setting. Finally, observe that every pair $(f, \eta_f) \in \Gamma^r F \times \Gamma_f$ is a section of $VF \stackrel{V\pi}{\longrightarrow} E$, where $V\pi = \pi_F \circ \tau_F|_{VF}$, and viceversa. Thus the space of sections $\Gamma^r(VF)$ can be considered as the full tangent bundle of $\Gamma^r F$.

Now consider a C^k PDO $\mathcal{L}_r$ and a section $\eta \in \Gamma^r_f$ and let f_t a 1-parameter smooth family of sections f_t of F defined as above so that

$$f_0 = f\,,\quad \left.\frac{df}{dt}\right|_{t=0} = \eta\,.$$

Definition 2.2.5. *The* linearization of $\mathcal{L}_r$ at f *is the linear PDO*

$$\ell_{r,f} : \Gamma^r_f \to \Gamma^0 G$$

defined by

$$\ell_{r,f}(\eta) = \left.\frac{d}{dt}\mathcal{L}_r(f_t)\right|_{t=0}$$

The PDO

$$\ell_r : \Gamma^r(VF) \to \Gamma^0 G\,.,$$

defined as $\ell_r(f,\eta) = \ell_{r,f}(\eta)$, is the tangent map *(or* differential*) of $\mathcal{L}_r$.*

A direct elementary calculation shows that this definition does not depend on the particular family f_t and therefore is well-posed. Often in this thesis we will use in calculations the notation δf, where δ stands for $d/dt|_{t=0}$, used often in mechanics and in the theory of calculus of variations, rather than η_f.

Example 2.2.6. *The Lie derivative L_ξ is linear and so it is to be expected that its differential ℓ_ξ is identical to it. Indeed*

$$\ell_\xi(f, \delta f) = \delta L_\xi(f) = \delta(\xi^\alpha \partial_\alpha f) = \xi^\alpha \partial_\alpha \delta f$$

The isometric operator $\mathcal{D}_{M,q}$ instead is quadratic and its differential $\ell_{M,q}$ is

$$\ell_{M,q}(f, \delta f) = \delta\mathcal{D}_{M,q}(f) = \delta(\delta_{ij}\partial_\alpha f^i \partial_\beta f^j) = 2\delta_{ij}\partial_\alpha f^i \partial_\beta \delta f^j$$

Now that we have defined the linearization of a PDO we can define its infinitesimal invertibility:

Definition 2.2.7. *We say that $\mathcal{L}_r$ is* infinitesimally invertible *over some subset $\mathcal{A} \subset \Gamma^r F$ if there exist a family of linear PDOs $m_f : \Gamma^s G \to \Gamma^0_f$ of some order s, with $f \in \mathcal{A}$, satisfying the following properties:*

1. *$\mathcal{A} \subset \Gamma^d F$ for some $d \geq r$ called* defect *of the infinitesimal inversion and $\mathcal{A}$ is equal to the set of sections f whose s-jets $j^s f$ are such that $j^s f(E) \subset A$ for some open subset $A \subset J^d F$;*

2. *the map $m : \mathcal{A} \times \Gamma^s G \to \Gamma^0(VF)$ is a PDO which is non-linear of order d in the first argument. Its corresponding jet spaces homomorphism is a map $A \times J^s G \to VF$;*

3. *$\ell_r(m(f, g)) = g$ for every $f \in \Gamma^{r+d} F$ and $g \in \Gamma^{r+s} G$.*

The most important example of infinitesimally invertible PDOs are the isometric operators, which were also the starting point of this whole theory. They will be discussed in next section. In the rest of the present section we state the Newton-Nash-Moser-Gromov Implicit Function Theorem, whose proof is a wide generalization of the original Nash proof of the inversion of the isometric operators.

Theorem 2.2.8. *Let $\mathcal{L}_r$ be a C^k PDO admitting an infinitesimal inverse of order s and defect d over some subset $\mathcal{A} \subset \Gamma^r F$ and set $\hat{s} = \max(d, 2r+s)+s+1$. Then, for every $f_0 \in \mathcal{A} \cap \Gamma^\infty F$, there is a neighbourhood $\mathcal{U} \subset \Gamma^{\hat{s}} G$ of 0 such that, for every $g \in \mathcal{U} \cap \Gamma^{s'}$ with $s' \geq \hat{s}$, the equation $\mathcal{L}_r(f) = \mathcal{L}_r(f_0) + g$ has a $C^{s'-s}$ solution.*

The following corollary is the version of the IFT most important for us:

Corollary 2.2.9. *Let $\mathcal{L}_r$ a PDO infinitesimally invertible over $\mathcal{A} \subset \Gamma^r F$. Then the restriction of $\mathcal{L}_r$ to $\mathcal{A} \cap \Gamma^\infty F$ is an open map.*

2.3 Free Maps and the Nash Theorem

As we already pointed out in Example 2.2.6, the linearization of the isometric operator

$$\begin{array}{rcl} \mathcal{D}_{M,q} : C^1(M, \mathbb{R}^q) & \longrightarrow & J^0(S^0_2 M) \\ (f^i) & \mapsto & \delta_{ij} \partial_\alpha f^i \partial_\beta f^j \, dx^\alpha \otimes dx^\beta \end{array}$$

is equal to

$$\ell_{M,q}(f, \delta f) = 2\delta_{ij} \partial_\alpha f^i \partial_\beta \delta f^j \, dx^\alpha \otimes dx^\beta \,.$$

Given a section $\delta g_{\alpha\beta} \in \Gamma^\infty(S^0_2 M)$, in order to use Gromov's IFT we must find some open set $\mathcal{A}$ of C^d functions from M to $\mathbb{R}^q$ over which the linear PDE system

$$2\delta_{ij} \partial_\alpha f^i \partial_\beta \delta f^j = \delta g_{\alpha\beta} \tag{2.1}$$

is solvable.

Following Nash, we take $f \in C^2(M, \mathbb{R}^q)$ and use the Leibniz rule to transform (2.1) in

$$2\delta_{ij}\left[\partial_\beta\left(\partial_\alpha f^i \delta f^j\right) - \partial^2_{\alpha\beta} f^i \delta f^j\right] = \delta g_{\alpha\beta} \tag{2.2}$$

and observe that, under the assumption

$$\delta_{ij}\partial_\alpha f^i \delta f^j = 0\,,$$

the system (2.2) is equivalent to (2.1). Hence for the solvability of (2.1) it is a sufficient condition the solvability of the larger system

$$\begin{cases} \delta_{ij}\partial_\alpha f^i \delta f^j &= 0 \\ \delta_{ij}\partial^2_{\alpha\beta} f^i \delta f^j &= -\delta g_{\alpha\beta}/2 \end{cases} \tag{2.3}$$

This justifies the following definition:

Definition 2.3.1. *A map $f \in C^2(M, \mathbb{R}^q)$ is said* free *if its first and second derivatives are linearly independent at each point, namely if, in coordinates, the $(m + s_m) \times q$ matrix*

$$\mathcal{D}^2 f = \begin{pmatrix} \partial_1 f^1 & \cdots & \partial_1 f^q \\ \vdots & \vdots & \vdots \\ \partial_m f^1 & \cdots & \partial_m f^q \\ \partial_{11} f^1 & \cdots & \partial_{11} f^q \\ \partial_{12} f^1 & \cdots & \partial_{12} f^q \\ \vdots & \vdots & \vdots \\ \partial_{mm} f^1 & \cdots & \partial_{mm} f^q \end{pmatrix}$$

has rank $m + s_m$ at every point. The set of all C^r free maps is denoted by $F^r(M, \mathbb{R}^q)$.

Clearly, since we endowed all functional spaces with the Whitney strong topology, the set $F^r(M, \mathbb{R}^q)$ is open in $C^r(M, \mathbb{R}^q)$ for it is defined by an open condition. Let $\mathcal{F}^2(M, \mathbb{R}^q)$ be the open subbundle of $J^2(M, \mathbb{R}^q) \xrightarrow{\pi} M \times \mathbb{R}^q$ whose fibers over each point (x^α, y^i) are the matrices $(y^i_\alpha, y^i_{\alpha\beta})$ of rank $m + s_m$. Then the free maps $f \in F^r(M, \mathbb{R}^q)$ are the functions $f \in C^r(M, \mathbb{R}^q)$ whose 2-jet satisfies the property $j^2 f(M) \subset \mathcal{F}^2(M, \mathbb{R}^q)$.

System (2.3) is clearly always solvable over $F^2(M, \mathbb{R}^q)$, leading to the following theorem:

Theorem 2.3.2 (Nash, 1956)**.** *The isometric operator $\mathcal{D}_{M,q}$ admits an infinitesimal inverse of defect 2 and order 0 over the space of free maps $F^2(M, \mathbb{R}^q)$.*

Proof. System (2.3) is linear and therefore gives us δf as an affine function of δg. The space of its solutions over every point f therefore is an affine subspace of the fiber of codimension $q - s_m - m$. Let $\delta f_{f,\delta g}$ be the solution closest to the origin with respect to the canonical euclidean metric on the fiber and observe that this solution is uniquely determined and depends smoothly on f because the coefficients of this affine subspace are regular if f is free.

Define now the operator $m(f, \delta g) = \delta f_{f,\delta g}$. This m is an infinitesimal inverse for $\mathcal{D}_{M,q}$ or order 0 and defect 2. Indeed by definition $l_{M,q}(m(f, \delta g)) = \delta g$, since $m(f, \delta g)$ is a solution of $l_{M,q}(f, \delta f) = \delta g$. Clearly m is order 0 with respect to δg since system (2.3) is purely algebraic. The defect of m is 2 because in order to solve (2.3) we must ask $f \in C^2(M, \mathbb{R}^q)$ since we need the matrix $\mathcal{D}^2 f$ to be continuous. Finally, as we already pointed out, free maps are characterized as sections of an open subbundle of $J^2(M, \mathbb{R}^q)$, so that all properties in Definition 2.3.1 are satisfied. $\square$

Theorems 2.3.2 and 2.2.8 immediately lead to the celebrated Nash theorem on C^k isometries with $k \geq 3$:

Theorem 2.3.3 (Nash, 1956)**.** *If $g_0 = \mathcal{D}_{M,q}(f_0)$ with $f_0 \in F^\infty(M, \mathbb{R}^q)$, then the C^s metric $g_0 + g$, $s \geq 3$, can be realized by a C^s immersion f (namely $\mathcal{D}_{M,q}(f) = g_0 + g$) for every C^3-small enough g.*

It is clear from the theorems above why free maps are a central concept in the theory of isometric immersions and embeddings. In the rest of the section we recall the main facts about these maps.

Proposition 2.3.4. *The set $F^r(M, \mathbb{R}^q)$ is empty for $q < m + s_m$ and dense (and in particular non-empty) for $q \geq 2m + s_m$.*

Proof. A map $f : M \to \mathbb{R}^q$ is free when the image of the map

$$D^2 f : M \to M_{s_m,q}(\mathbb{R})$$

is contained in the set of matrices of maximal rank. In particular a map is *not* free when the image of $D^2 f$ intersects the set $\mathcal{N}_{s_m,q}$ of matrices of non-maximal rank, whose codimension is $q - s_m + 1$ [24]. For a generic f the image $D^2 f(M)$ and $\mathcal{N}_{s_m,q}$ are transversal and therefore they do not have points in common when $\dim D^2 f(M) < \operatorname{codim} \mathcal{N}_{s_m,q}$. Hence a generic map f is free for $q \geq m + s_m$. $\square$

Following Gromov, we call "extra-dimension" the cases with $q \geq m + s_m$ and "critical dimension" the case $q = m + s_m$.

Theorem 2.3.5 (h-principle for free maps)**.** *Free maps $M \to \mathbb{R}^q$ satisfy the h-principle in the extra dimension case and, if M is open (i.e. has no compact component), in the critical dimension case.*

We recall that this means simply that C^r free maps arise between M and $\mathbb{R}^q$ iff the bundle $\mathcal{F}^2(M, \mathbb{R}^q) \xrightarrow{\pi} M \times \mathbb{R}^q$ admits a C^r section. In particular then if M is parallelizable then $F^r(M, \mathbb{R}^q)$, $r \geq 2$, is non-empty for all $q \geq m + s_m$ if M is open and for all $q \geq m + s_m + 1$ if M has a compact component.

Example 2.3.6. *The sets $F^r(\mathbb{R}^m, \mathbb{R}^q)$ are non-empty for all $q \geq m + s_m$ The simplest free maps in the critical dimensions between euclidean spaces are*

$$f(x^\alpha) = (x^1, \dots, x^m, (x^1)^2, x^1x^2, \dots, (x^m)^2)$$

and its compositions with the permutations of $m + s_m$ variables.

It is interesting to notice that, out of this map, one can extract the free embedding $V : \mathbb{R}\mathrm{P}^m \to \mathbb{R}^q$, again in the critical dimension $q = m + s_m$ and introduced first by Veronese, given by

$$V([x^1 : \cdots : x^{m+1}]) = [(x^1)^2 : x^1x^2 : \cdots : (x^{m+1})^2],$$

which embeds $\mathbb{R}\mathrm{P}^m$ in some affine q-plane inside $\mathbb{R}\mathrm{P}^q$. The lift of this map to $\mathbb{S}^m$ provides a free map in critical dimension for all spheres. These are the only compact manifolds for which it is known there exist free maps in critical dimension.

Example 2.3.7. *The sets $F^r(\mathbb{T}^m, \mathbb{R}^q)$ are non-empty at least for all $q \geq m + s_m + 1$. For example, the map $f : \mathbb{T}^2 \to \mathbb{R}^6$ defined by*

$$f(x, y) = (\cos x, \sin x, \cos y, \sin y, \cos(x + y), \sin(x + y))$$

belongs to $F^\infty(\mathbb{T}^2, \mathbb{R}^6)$.

2.4 Algebraic Solution of under-determined linear PDEs

In this section we illustrate in detail some results of Gromov on under-determined linear PDE systems by translating most of Chapter 2, Section 3, Subsection 8 of [1] in our notations and using a style less compact than the original.

We assume throughout the section that $F \xrightarrow{\pi_F} E$ is a vector bundle, so that its fibers over any point $e_0 \in E$ are (non-canonically) isomorphic to $\mathbb{R}^q$. Recall that also $G \xrightarrow{\pi_G} E$ is a vector bundle whose fibers are (non-canonically) isomorphic to $\mathbb{R}^{q'}$. A linear PDO of order r is a linear map $\mathcal{L}_r : \Gamma^r F \to \Gamma^0 G$. The corresponding map $\Lambda_r : J^r F \to J^0 G$ is a homomorphism of vector bundles.

Definition 2.4.1. *We say that $\mathcal{L}_r$ is* under-determined *if $q < q'$.*

If $\mathcal{L}_r$ is under-determined then the linear PDE $\mathcal{L}_r f = g$ has more unknowns than equations. The goal of this section is to present in detail Gromov's argument that shows that a generic under-determined PDE is solvable.

Before going to the general case we illustrate a few elementary cases.

2.4.1 Operators with constant coefficients

Consider the case $M = \mathbb{R}^m$ and an operator $\mathcal{L}_r : C^r(\mathbb{R}^m, \mathbb{R}^q) \to C^0(\mathbb{R}^m, \mathbb{R}^{q'})$ with constant coefficients, namely

$$\mathcal{L}_r(f) = \left(\sum_{|A| \leq r} \Lambda_i^{aA} \partial_A f^i \right) = \left(\Lambda_i^a f^i + \Lambda_i^{a\alpha} \partial_\alpha f^i + \cdots + \Lambda_i^{a\alpha_1 \dots \alpha_r} \partial_{\alpha_1 \dots \alpha_r} f^i \right)$$

for some constant matrices Λ_i^{aA}. In this particular case we can define PDOs in a further equivalent way:

Definition 2.4.2. *A C^k PDO of order r between $C^r(\mathbb{R}^m, \mathbb{R}^q)$ and $C^r(\mathbb{R}^m, \mathbb{R}^{q'})$ is a $q \times q'$ matrix $\mathfrak{L}_r = (\mathfrak{L}_i^a)$ whose elements $\mathfrak{L}_i^a = \Lambda_i^{aA}\partial_A$ are C^k PDOs of $C^r(\mathbb{R}^m)$ in itself.*

A right inverse for $\mathcal{L}_r$ is an operator $\mathcal{M}_s : C^{r+s}(\mathbb{R}^m, \mathbb{R}^{q'}) \to C^r(\mathbb{R}^m, \mathbb{R}^q)$ such that $\mathcal{L}_r \circ \mathcal{M}_s = i_0^{r+s}$, where i_0^{r+s} is the canonical injection $C^{r+s}(\mathbb{R}^m, \mathbb{R}^{q'}) \to C^0(\mathbb{R}^m, \mathbb{R}^{q'})$.

Assume first that $q' = 1$ and that $M_s = \sum_{|B| \leq s} M^{iB}\partial_B$ has also constant coefficients. Then to each matrix Λ_i^A and M^{iB} we can associate polynomials of degrees respectively r and s in $\mathbb{C}^m$ defined by

$$\hat{\Lambda}_i(w^1, \ldots, w^m) = \Lambda_i + \Lambda_i^\alpha w_\alpha + \cdots + \Lambda_i^{\alpha_1 \ldots \alpha_r} w_{\alpha_1} \cdots w_{\alpha_r}$$

$$\hat{M}^i(w^1, \ldots, w^m) = M^i + M^{i\alpha} w_\alpha + \cdots + M^{i\alpha_1 \ldots \alpha_s} w_{\alpha_1} \cdots w_{\alpha_s}$$

The relation $\mathcal{L}_r \circ \mathcal{M}_s = \sum_{|A| \leq r} \sum_{|B| \leq s} \Lambda_i^{aA} M^{iB} \partial_{AB} = i_0^{r+s}$ translates in

$$\hat{\Lambda}_i \hat{M}^i = 1 \tag{2.4}$$

namely the ideal generated by the $\hat{\Lambda}_i$ is the whole ring $\mathbb{C}[w_1, \ldots, w_m]$ of complex polynomials in m variables. Clearly (2.4) holds iff the system

$$\begin{cases} \hat{\Lambda}_1(w^1, \ldots, w^m) = 0 \\ \vdots \\ \hat{\Lambda}_q(w^1, \ldots, w^m) = 0 \end{cases} \tag{2.5}$$

admits no solution and this, for a set of q generic polynomials, can happen iff $q > m$. Thus $\mathcal{L}_r$, with $q' = 1$, is generically invertible for $q \geq m + 1$.

For $q' > 1$, to the operator $\mathcal{L}_r$ we can associate a $m \times q'$ matrix $(\hat{\Lambda}_i^a)$ of polynomials and similarly for $\mathcal{M}_s$ and the invertibility condition writes

$$\hat{\Lambda}_i^a \hat{M}_b^i = \delta_b^a \,.$$

This can happen iff $\mathrm{rank}(\hat{\Lambda}_i^a) = q'$, which is represented by $q' + m - 1$ equations. Hence such a generic $\mathcal{L}_r$ is invertible iff $q \geq q' + m$.

2.4.2 Lie Equations

Consider a finite set $X_q = \{\xi_1, \ldots, \xi_q\}$ of vector fields $\xi_i \in \mathfrak{X}(M)$ and the differential operator $\mathcal{D}_{X_q} : C^\infty(M, \mathbb{R}^q) \to C^\infty(M)$ defined by

$$\mathcal{D}_{X_q}(f^1, \ldots, f^q) = \begin{pmatrix} L_{\xi_1} & \ldots & L_{\xi_q} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_q \end{pmatrix} = L_{\xi_i} f^i$$

Definition 2.4.3. *We say that X_q is* large *if there exist q smooth functions λ^i such that $\lambda^i \xi_i = 0$ and $\xi_i(\lambda^i) = \mu \in C^\infty_+(M)$.*

Large collections of vector fields are interesting because the corresponding PDE $\mathcal{D}_{X_q}(f^1, \dots, f^q) = g$, restricted to a suitable subspace, becomes algebraic:

Proposition 2.4.4. *If X_q is large then the restriction of the PDE*

$$\mathcal{D}_{X_q}(f^1, \dots, f^q) = g$$

to the subspace $\mathcal{A}_q = \{h(\lambda^1, \dots, \lambda^q) \,|\, h \in C^\infty(M)\} \subset C^\infty(M, \mathbb{R}^q)$ is purely algebraic.

Proof. Set $f^i = h\lambda^i$. Then

$$\mathcal{D}_{X_q}(f^1, \dots, f^q) = L_{\xi_i}(h\lambda^i) = hL_{\xi_i}\lambda^i + \lambda^i L_{\xi_i} h = h\mu + L_{\lambda^i \xi_i} h = h\mu\,,$$

namely the equation $\mathcal{D}_{X_q}(f\lambda^1, \dots, f\lambda^q) = g$ is equivalent to the algebraic equation $\mu h = g$. □

Theorem 2.4.5. *If X_q is large then $\mathcal{D}_{X_q}$ is surjective.*

Proof. This is just due to the fact that every $\mu \in C^\infty_+(M)$ has a smooth inverse. Hence the restriction of $\mathcal{D}$ to $\mathcal{A}_q$ is surjective by Proposition 2.4.4 and a solution to $\mathcal{D}_{X_q}(f^1, \dots, f^q) = g$ is given, for every $g \in C^\infty(M)$, by $f^i = \lambda^i\, g/\mu$. □

Theorem 2.4.6. *A generic X_q is large for $q \geq 2m + 1$.*

Proof. In order for X_q to be large we must be able to solve the algebro-differential system

$$\begin{cases} \lambda^i \xi_i^\alpha = 0 \\ \xi_i^\alpha \partial_\alpha \lambda^i = g \end{cases} \tag{2.6}$$

of $m + 1$ equations in q unknowns. The second (differential) set of equations transforms into algebraic ones by observing that

$$\xi_i^\alpha \partial_\alpha \lambda^i = \partial_\alpha(\lambda^i \xi_i^\alpha) - \lambda^i \partial_\alpha \xi_i^\alpha\,,$$

so that system (2.6) is equivalent to the linear system

$$(\,\lambda^1 \,\cdots\, \lambda^q\,) \begin{pmatrix} \xi_1^1 & \cdots & \xi_1^m & \partial_\alpha \xi_1^\alpha \\ \vdots & \vdots & \vdots & \vdots \\ \xi_q^1 & \cdots & \xi_q^m & \partial_\alpha \xi_q^\alpha \end{pmatrix} = (\overbrace{0 \,\cdots\, 0}^{m}\; -g\,)\,. \tag{2.7}$$

If X_q is generic then the matrix

$$D_{X_q} = \begin{pmatrix} \xi_1^1 & \cdots & \xi_1^m & \partial_\alpha \xi_1^\alpha \\ \vdots & \vdots & \vdots & \vdots \\ \xi_q^1 & \cdots & \xi_q^m & \partial_\alpha \xi_q^\alpha \end{pmatrix}$$

can be considered a generic application $D_{X_q} : M \to \mathcal{M}_{(m+1)\times q}(\mathbb{R})$. The space of real $(m+1) \times q$ matrices of non-full rank have codimension $q - m$ (e.g. see [24]) and therefore the map D_{X_q} does not intersect it if $m < q + n$, namely if $q > 2m$. $\square$

As the following example shows, the condition $q \geq 2m + 1$ is not at all necessary for the existence of large collections of vector fields:

Example 2.4.7. *The set $X_{m+1} = \{\partial_1, \dots, \partial_n, x^\alpha \partial_\alpha\}$ is large. Indeed for this case we can set $\lambda^\alpha = x^\alpha$, $\alpha = 1, \dots, m$, $\lambda^{m+1} = -1$.*

Definition 2.4.8. *A Lie subalgebra of $\mathfrak{X}(M)$ is* large *if it contains a large collection of vector fields. We say that $X_q \subset \mathfrak{X}(M)$ is* weakly large *if the Lie algebra $\langle X_q \rangle \subset \mathfrak{X}(M)$ generated by it is large.*

Theorem 2.4.9. *If X_q is weakly large then $\mathcal{D}_{X_q}$ is surjective.*

Proof. Let $Y_p = \{\eta_1, \dots, \eta_p\}$ a large subset of the Lie algebra generated by X_q. By definition, every operator L_{η_k} is equal to a sum $L_{\xi_i} \Xi^i_k$, where Ξ^i_k is some PDO of finite order; for example, $L_{[\xi_1,\xi_2]} = L_{\xi_1} \Xi^1 + L_{\xi_2} \Xi^2$ for $\Xi^1 = L_{\xi_2}$ and $\Xi^2 = L_{\xi_1}$. Therefore for every $g \in C^\infty(M)$, by hypothesis, we can find p functions F^k s.t. $L_{\eta_k} F^k = g$, namely $L_{\xi_i} \Xi^i_k F^k = g$, so that the functions $f^i = \Xi^i_k F^k$ solve $L_{\xi_i} f^i = g$, i.e. $\mathcal{D}_{X_q}$ is surjective. $\square$

Example 2.4.10. *The set $X_2 = \{\partial_x, (x+y)\partial_y\} \subset \mathfrak{X}(\mathbb{R}^2)$ is not large but it is weakly large. Indeed $[\partial_x, (x+y)\partial_y] = \partial_y$ and*

$$X_3 = \{\xi_1 = \partial_x, \xi_2 = \partial_y, \xi_3 = (x+y)\partial_y\} \subset \langle X_2 \rangle$$

is already large. For example we can take $\lambda^1 = 0$, $\lambda^2 = x + y$ and $\lambda^3 = -1$, so that

$$\lambda_i \xi^i = 0\partial_x + (x+y)\partial_y - (x+y)\partial_y = 0$$

and

$$\xi^i(\lambda_i) = \partial_x 0 + \partial_y(x+y) - (x+y)\partial_y(-1) = 1\,.$$

Proposition 2.4.11. *Any two vector fields $\xi_1, \xi_2 \in \mathfrak{X}(M)$ in generic position are weakly large.*

Proof. If ξ_1 and ξ_2 are a pair of generic vector fields then no linear relation occurs between them and their commutators. Consider the set Y_{2m+1} of any $2m+1$ of them. The elements of the matrix $D_{Y_{2m+1}}$ then can be considered independent and therefore Y_{2m+1} is large. $\square$

Corollary 2.4.12. *If $\xi_1, \xi_2 \in \mathfrak{X}(M)$ are generic, the PDO*

$$\begin{pmatrix} L_{\xi_1} & L_{\xi_2} & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_q \end{pmatrix} = L_{\xi_1} f^1 + L_{\xi_1} f^2$$

is surjective.

2.4.3 The general case

Consider finally a general linear PDO $\mathcal{L}_r : \Gamma^r F \to \Gamma^0 G$ of order r. In this case

$$\mathcal{L}_r(f) = \left(\sum_{|A| \le r} \Lambda_i^{aA} \partial_A f^i \right) = \left(\Lambda_i^a f^i + \Lambda_i^{a\alpha} \partial_\alpha f^i + \cdots + \Lambda_i^{a\alpha_1 \ldots \alpha_r} \partial_{\alpha_1 \ldots \alpha_r} f^i \right)$$

for some linear homomorphisms $\Lambda_r : J^r F \to G$. This suggests the following equivalent definition for a linear PDO:

Definition 2.4.13. *A linear C^k PDO over F of order r is a C^k section Λ_r of the bundle $Hom(J^r F, G) \to E$ of all linear homomorphisms between $J^r F$ and some vector bundle G.*

In every trivialization of F and G, Λ_r can be represented by a PDO matrix $\mathfrak{L}_r$ as in case of the linear PDOs with constant coefficients but this representation is not global in general.

To every $\mathcal{L}_r$ we can associate the adjoint operator $\mathcal{L}_r^* : \Gamma^r G \to \Gamma^0 F$ defined by

$$\mathcal{L}_r^* g = \left(\sum_{|A| \le r} (-1)^{|A|} \partial_A \left(\overline{\Lambda}_a^{iA} g^a \right) \right) ,$$

where $\overline{\Lambda}_a^{iA} = \Lambda_i^{aA}$ is the transpose matrix and $\Lambda_r^* = (\overline{\Lambda}_a^{iA}) : J^r G \to F$. Note that the higher order terms of an operator and its adjoint are exactly the transposed of each other, while the terms of lower order are mixed in a more complicated way.

Example 2.4.14. *Consider the case of a first-order operator $\mathcal{L}_1 : \Gamma^1 F \to \Gamma^0 G$, so that*

$$\mathcal{L}_1(f) = \left(\Lambda_i^a f^i + \Lambda_i^{a\alpha} \partial_\alpha f^i \right) .$$

Its adjoint $\mathcal{L}_1^* : \Gamma^1 G \to \Gamma^0 F$ is the operator

$$\mathcal{L}_1^*(g) = \left(\overline{\Lambda}_a^i g^a - \partial_\alpha (\overline{\Lambda}_a^{i\alpha} g^a) \right) = \left((\overline{\Lambda}_a^i - \partial_\alpha \overline{\Lambda}_a^{i\alpha}) g^a - \overline{\Lambda}_a^{i\alpha} \partial_\alpha g^a \right) .$$

Example 2.4.15. *The operator L_ξ (see Example 2.2.3) is linear. The $m+1$ coefficients of the homomorphisms Λ_ξ are*

$$(\Lambda_\xi)_1^1 = 0 \,, \; (\Lambda_\xi)_1^{1\alpha} = \xi^\alpha \,.$$

A direct calculation shows that, as expected, $L_\xi^* = -L_\xi$.

Example 2.4.16. *The homomorphism $\Lambda_{M,q} : J^1(M, \mathbb{R}^q) \to J^0(S_2^0 M)$ corresponding to the differential $\ell_{M,q}$ of the isometric operator $\mathcal{D}_{M,q}$ (see Example 2.2.6) writes*

$$\ell_{M,q}(f, \delta f) = (\Lambda_{M,q})^\gamma_{(\alpha\beta)i}(f) \, \partial_\gamma \delta f^i \, dx^\alpha \otimes dx^\beta ,$$

where by $(\alpha\beta)$ we denote the coordinates of a section $\delta g_\alpha\beta$ of S^0_2M, to distinguish them from the index γ which also run from 1 to m but is instead contracted with the derivative ∂_γ. Its coefficients are given by

$$\begin{cases} (\Lambda_{M,q})_{(\alpha\beta)i} &= 0 \\ (\Lambda_{M,q})^{\gamma}_{(\alpha\beta)i} &= \delta_{ij}\left[\delta^\gamma_\alpha \partial_\beta f^j + \delta^\gamma_\beta \partial_\alpha f^j\right] \end{cases} . \tag{2.8}$$

The coefficients of the adjoint homomorphism $\Lambda^*_{M,q} : J^0(S^0_2M) \to J^1(M, \mathbb{R}^q)$ associated to $\ell^*_{M,q}$ are

$$\begin{cases} (\Lambda^*_{M,q})^{(\alpha\beta)i} = 2\,\partial_{\alpha\beta} f^i \\ (\Lambda^*_{M,q})^{(\alpha\beta)i\gamma} = -\delta^\gamma_\alpha\,\partial_\beta f^i + \delta^\gamma_\beta\,\partial_\alpha f^i \end{cases} , \tag{2.9}$$

so that

$$\begin{aligned} \ell^*_{M,q}(f, \delta g) &= \Big((\Lambda^*_{M,q})^{(\alpha\beta)i}(f)\,\delta g_{\alpha\beta} + (\Lambda_{M,q})^{(\alpha\beta)i\gamma}(f)\,\partial_\gamma \delta g_{\alpha\beta}\Big) \\ &= \Big(2\,\partial_{\alpha\beta} f^i\,\delta g_{\alpha\beta} - \delta^\gamma_\alpha\,\partial_\beta f^i + \delta^\gamma_\beta\,\partial_\alpha f^i\,\delta g_{\alpha\beta}\Big) . \end{aligned}$$

A direct calculation shows that the adjoint operation satisfies the expected properties

1. $(\mathcal{L}^*_r)^* = \mathcal{L}_r$;
2. $(\mathcal{L}_r\mathcal{M}_s)^* = \mathcal{M}^*_s\mathcal{L}^*_r$.

For dimensional reasons there cannot be a left inverse for $\mathcal{L}_r$ but there can be a *right* inverse, i.e. a linear PDO

$$\mathcal{M}_s : \Gamma^s G \to \Gamma^0 F$$

such that

$$\mathcal{L}_r\mathcal{M}_s = i^0_{s+r}(G) ,$$

namely

$$\mathcal{L}_r\mathcal{M}_s : \Gamma^{r+s}G \to \Gamma^0 G \text{ and } \mathcal{L}_r\mathcal{M}_s(g) = g .$$

In principle this fact could be useless since the equation $\mathcal{L}_r\mathcal{M}_s = i^0_{s+r}(G)$ is a rather complex PDE of order r in the elements of $\mathcal{M}_s$ having for coefficients linear functions of the components of $\mathcal{L}_r$. The reason why it is instead of fundamental importance is that, on the contrary, the equivalent equation

$$\mathcal{M}^*_s\mathcal{L}^*_r = i^0_{s+r}(G)$$

is *linear* in the elements of $\mathcal{M}_s$ and the coefficients of this linear system of equations depend on the elements of $\mathcal{L}_r$ and on their derivatives up to order s.

In terms of jets, to the PDO $\mathcal{M}_s$ correspond a vector bundle morphism $M_s : J^sG \to F$ which is the inverse of Λ_r in the sense that

$$j^r(M_s g)^*\Lambda_r = g , \ \forall g \in \Gamma^{s+r}(G)$$

The adjoint version of this equation reads

$$j^s(\Lambda_r^* g)^* M_s^* = g \, , \; \forall g \in \Gamma^{s+r}(G)$$

which in local coordinates writes

$$\overline{M}_i^{aA} \partial_A \left[\overline{\Lambda}_b^{iB} \partial_B \right] = \delta_b^a$$

or, more explicitly and after dropping the bar over the elements of M_s^* and Λ_r^* to make notation lighter,

$$\begin{cases} \displaystyle\sum_{|A|\le s} M_i^{aA} \partial_A \Lambda_b^i = \delta_b^a \\ \displaystyle\sum_{|A|\le s} M_i^{aA} \partial_A \Lambda_b^{i\beta_1} + \sum_{|A|\le s-1} M_i^{a\beta_1 A} \partial_A \Lambda_b^i = 0 \\ \displaystyle\sum_{|A|\le s} M_i^{aA} \partial_A \Lambda_b^{i\beta_1\beta_2} + \sum_{|A|\le s-1} M_i^{a\beta_1 A} \partial_A \Lambda_b^{i\beta_2} + \sum_{|A|\le s-2} M_i^{a\beta_1\beta_2 A} \partial_A \Lambda_b^i = 0 \\ \vdots \\ \displaystyle\sum_{|A|\le 1} M_i^{a\beta_1\dots\beta_{s-1}A} \partial_A \Lambda_b^{i\beta_s\dots\beta_{s+r-1}} + M_i^{a\beta_1\dots\beta_s} \Lambda_b^{i\beta_{s+1}\dots\beta_{s+r-1}} = 0 \\ M_i^{a\beta_1\dots\beta_s} \Lambda_b^{i\beta_{s+1}\dots\beta_{s+r}} = 0 \end{cases} \tag{2.10}$$

This huge linear system in the $qq'\binom{m+s}{s}$ unknowns M_i^{aA} consists of $(q')^2$ equations at the order 0 (i.e. containing the terms of $\mathcal{M}_s^* \mathcal{L}_r^*$ of order 0), $(q')^2 m$ equations at order 1 and so on up to the order $s+r$, consisting of $(q')^2 \binom{m+r+s-1}{r+s}$ equations, for a total of $(q')^2 \binom{m+r+s}{r+s}$ equations.

In particular the unknowns are more than the equations when

$$q\frac{(m+s)!}{s!} > q'\frac{(m+r+s)!}{(r+s)!}$$

namely

$$\frac{q}{q'} > \frac{(m+r+s)!}{(m+s)!}\frac{s!}{(r+s)!} = \frac{m+s+r}{s+r} \cdots \frac{m+s+1}{s+1} = \prod_{i=1}^{r}(1+\frac{m}{s+i})$$

For example this surely happens when

$$\frac{q}{q'} > \left(1+\frac{m}{s}\right)^r , \tag{2.11}$$

from which it is clear that, as long as $q > q'$, it is always possible to choose $\mathcal{M}_s$ of order s so high to satisfy the inequality. We are going to show below that this condition is actually sufficient for the (formal) solvability of the system.

As a first step toward solving (2.10) we observe that it naturally splits q' independent systems, each of them obtained by keeping only those equations containing the unknowns $M_i^{a_0 A}$ for some fixed a_0, since in no equation appear at the same time unknowns with two different values for that index. Each of these systems has $q'\binom{m+r+s}{r+s}$ equations and within each of them only one equation, precisely

$$\sum_{|A|\leq s} M_i^{a_0 A}\partial_A \Lambda^i_{a_0} = 1\,, \tag{2.12}$$

has a rhs different from 0.

Observe now that, since F is a vector bundle and Λ_r is a linear morphism of bundles, Λ_r and Λ_r^* can be seen respectively as sections of the bundles $\mathrm{Hom}(J^r F, G) \to E$ and $\mathrm{Hom}(J^r G, F) \to E$ of all such morphisms. Since the elements Λ_a^{iA} of Λ_r^* are functions of the elements Λ_i^{aA} of Λ_r and of their derivatives up to order r, the coefficients of system (2.10) are functions of the Λ_i^{aA} and their derivatives up to order $r+s$. It is more convenient though for us to consider the elements of the adjoint as independent variables, in particular as coordinates on the fibers of $J^s\,\mathrm{Hom}(J^r G, F) \to E$. The dimension of these fiber is $qq'\binom{m+r}{r}\binom{m+s}{s}$. In $J^s\,\mathrm{Hom}(J^r G, F)$ we denote by Λ^{iB}_{bA}, with $|A|\leq s$ and $|B|\leq r$, the coordinates corresponding to the partial derivatives $\partial_A \Lambda_b^{iB}$, so that system (2.10), with $a = a_0$, writes as

$$\begin{cases} \displaystyle\sum_{|A|\leq s} M_i^{a_0 A}\Lambda^i_{bA} = \delta^a_b \\ \displaystyle\sum_{|A|\leq s} M_i^{a_0 A}\Lambda^{i\beta_1}_{bA} + \sum_{|A|\leq s-1} M_i^{a_0\beta_1 A}\Lambda^i_{bA} = 0 \\ \displaystyle\sum_{|A|\leq s} M_i^{a_0 A}\Lambda^{i\beta_1\beta_2}_{bA} + \sum_{|A|\leq s-1} M_i^{a_0\beta_1 A}\Lambda^{i\beta_2}_{bA} + \sum_{|A|\leq s-2} M_i^{a_0\beta_1\beta_2 A}\Lambda^i_{bA} = 0 \\ \vdots \\ \displaystyle\sum_{|A|\leq 1} M_i^{a_0\beta_1\ldots\beta_{s-1}A}\Lambda^{i\beta_s\ldots\beta_{s+r-1}}_{bA} + M_i^{a_0\beta_1\ldots\beta_s}\Lambda_b^{i\beta_{s+1}\ldots\beta_{s+r-1}} = 0 \\ M_i^{a_0\beta_1\ldots\beta_s}\Lambda_b^{i\beta_{s+1}\ldots\beta_{s+r}} = 0 \end{cases} \tag{2.13}$$

whose only non-homogeneous row is

$$\sum_{|A|\leq s} M_i^{a_0 A}\Lambda^i_{a_0 A} = 1\,. \tag{2.14}$$

Clearly the only obstruction to the existence of a formal solution of system (2.13) is that the non-homogeneous row (2.14) be a linear combination of the remaining rows with coefficients λ_k, $k = 1, \ldots, q'\binom{m+s+r}{s+r} - 1$, in the field $\mathcal{R}$ of rational functions in the fiber coordinates Λ^{iB}_{aA}. Let us assume, by absurd, that such a linear combination exists and observe that the system (2.13) is somehow "triangular", in the sense that the variable $\Lambda^i_{a_0 A}$ does not appear on the left or on the same column with respect to the column where it appears in the non-homogeneous row (2.14). Then we can do the following: starting with the leftmost coefficients $\Lambda^i_{a_0}$ we write them as linear combinations of the coefficients lying in the same column and then substitute this expression in all columns at their right, so that in the rest of the system the $\Lambda^i_{a_0}$ will not appear anymore. We do this recursively for each $\Lambda^i_{a_0 A}$ so that, in the end, we are left with relations

$$\Lambda^i_{a_0 A} = \Phi^i_{a_0 A}(\lambda_1, \dots, \lambda_k, \Lambda^{iB}_{aA}), \tag{2.15}$$

where the $\Phi^i_{a_0 A}$ are linear functions of the Λ^{iB}_{aA} and polynomial functions of the coefficients λ_k and satisfy $\partial_{\Lambda^i_{a_0 A}} \Phi^{i'}_{a_0 A'} = 0$ for all A, A', i and i' (i.e. no $\Lambda^i_{a_0 A}$ appears explicitly in the rhs of (2.15)). Since the $q\binom{m+s}{s}$ variables $\Lambda^i_{a_0 A}$, as coordinates on the fibers of $J^s \operatorname{Hom}(J^r G, F)$, are clearly independent, their number cannot be larger than the number $q'\binom{m+s+r}{s+r} - 1$ of homogeneous rows. Indeed, as no $\Lambda^i_{a_0 A}$ appears explicitly in the functions $\Phi^{i'}_{a_0 A'}$, in order for the relations (2.15) to hold they must be contained inside the coefficients λ_k and so the λ_k must be at least as many as the $\Lambda^i_{a_0 A}$. This shows that a necessary condition for the non-homogeneous row to be a linear combination of the homogeneous ones is that

$$q'\binom{m+s+r}{s+r} > q\binom{m+s}{s}.$$

Hence it is enough to ask the opposite inequality to ensure the solvability of (2.13). In particular, for the system to be (formally) solvable it is enough (see (2.11)) taking

$$s > \frac{n}{\left(\dfrac{q}{q'}\right)^{\frac{1}{r}} - 1}.$$

We assume from now on that s is chosen big enough to grant the existence solutions of (2.10). Such solutions will express the elements M^{aA}_i as rational functions of the $\partial_A \Lambda^{iB}_a$. Let M^*_s be one of these solutions and let $\mathfrak{p}$ be the polynomial in Λ^{iB}_{aA} obtained as the product of all denominators of its coefficients M^{aA}_i. Then the PDO $\mathcal{P}^*_s = \mathfrak{p}\mathcal{M}^*_s$ is polynomial in the $\partial_A \Lambda^{iB}_a$ and satisfies

$$\mathcal{P}^*_s \mathcal{L}^*_r = \mathfrak{p}\, i^0_{r+s}(G).$$

Clearly $\mathcal{M}^*_s$ is not regular in the zero set $Z_\mathfrak{p}$ of $\mathfrak{p}$. Let N^*_s be a second distinct solution, $\mathcal{N}^*_s$ the corresponding PDO, $\mathfrak{q}$ the corresponding polynomial product of all denominators of its coefficients and $Z_\mathfrak{q}$ its zero set. Out of $\mathcal{M}_s$ and $\mathcal{N}_s$ we can build a new, more regular, left inverse for $\mathcal{L}^*_r$. Indeed let $\lambda_\mathfrak{p}$, $\lambda_\mathfrak{q}$ be a pair of non-negative functions such that $\lambda_\mathfrak{p} + \lambda_\mathfrak{q} = 1$, $\lambda_\mathfrak{p}|_{Z_\mathfrak{p}} = 0$ and $\lambda_\mathfrak{q}|_{Z_\mathfrak{q}} = 0$. The operator $\lambda_\mathfrak{p}\mathcal{M}_s + \lambda_\mathfrak{q}\mathcal{N}_s$ is clearly a left inverse of $\mathcal{L}^*_r$ of order at most s and it is regular everywhere except on $Z_\mathfrak{p} \cap Z_\mathfrak{q}$. Note that, if $\mathfrak{p}$ and $\mathfrak{q}$ have common factors, then the codimension of $Z_\mathfrak{p} \cap Z_\mathfrak{q}$ remains 1 rather than dropping to 2. We say that $\mathcal{M}_s$ and $\mathcal{N}_s$ are functionally dependent or independent according to whether the corresponding polynomials are, so that when they are functionally independent the codimension of $Z_{\mathfrak{p}} \cap Z_{\mathfrak{q}}$ amounts to 2.

We can repeat these considerations for every solution of (2.10) so that, in the end, we can build a left inverse of $\mathcal{L}^*_r$ of order s which is regular at every point except at those belonging to the zero set Σ_s of the ideal $\mathfrak{P}_s$ of all polynomials $\mathfrak{p}$ such that $\mathcal{P}^*_s\mathcal{L}^*_r = \mathfrak{p}\, i^0_{r+s}(G)$ for some operator $\mathcal{P}_s$ with polynomial coefficients in the componentns of $j^s\Lambda^*_r$, namely the $\partial_A\Lambda^{iB}_a$. The codimension of Σ_s, i.e. the codimension of its irreducible component of higher codimension, is given therefore by the smallest number of functionally independent left inverses of $\mathcal{L}^*_r$. Finally observe that if $\mathfrak{p} \in \Sigma_s$, so that $\mathcal{P}^*_s\mathcal{L}^*_r = \mathfrak{p}\, i^0_{s+r}(G)$ for some polynomial operator $\mathcal{P}^*_s$, then also $\mathfrak{p} \in \Sigma_{s'}$ for every $s' \geq s$ since, trivially, $\mathcal{P}^*_{s'}\mathcal{L}^*_r = \mathfrak{p}\, i^0_{s'+r}(G)$, where $\mathcal{P}^*_{s'}$ has all coefficients of order up to s equal to those of $\mathcal{P}^*_s$ and all others equal to 0. In particular this means that $\pi^s_{s'}(\Sigma_{s'}) = \Sigma_s$ for the canonical projection $\pi^s_{s'} : J^{s'}\operatorname{Hom}(J^rG, F) \to J^s\operatorname{Hom}(J^rG, F)$.

Hence, in order to determine whether generic PDOs $\mathcal{L}_r$ admit a left inverse we must evaluate the number of its functionally independent left inverses. The following clever argument of Gromov settles the problem by showing how to build, as long as the codimension k of Σ_s is not larger than m, a new left inverse of some order $s' > s$ out of the ones of order s and functionally independent on them, so that $\Sigma_{s'}$ has codimension at least $k+1$; of course this ultimately implies that codim $\Sigma_s > m$ for s big enough and therefore that generic linear PDOs are in fact left-invertible (and therefore surjective on their target space) for $q > q'$.

Consider an irreducible component Σ_0 of Σ_s of codimension k, let x_0 be any regular point of Σ_0 and let $\mathfrak{p}_1, \dots, \mathfrak{p}_k : J^s\operatorname{Hom}(J^rG, F) \to \mathbb{R}$ be polynomials vanishing on Σ_0 and functionally independent at x_0. Since m is much smaller than the dimension of the fibers of $J^s\operatorname{Hom}(J^rG, F)$, a generic section $\Lambda^*_r : E \to \operatorname{Hom}(J^rG, F)$ is such that $j^s\Lambda^*_r(E)$ cuts Σ_0 in a set of dimension $m - k$ and the jacobian of $j^s\Lambda^*_r$ has rank m. In particular we can always pick a Λ_r passing through x_0 and slightly perturb it so that:

1. it cuts Σ_0 at $j^s\Lambda^*_r(e_0)$ close to x_0 for some $e_0 \in E$;
2. its tangent map is injective on the tangent of Σ_0, i.e. the k functions $(j^s\Lambda^*_r)^*\mathfrak{p}_i : E \to \mathbb{R}$ are functionally independent at e_0;
3. it has no characteristic submanifold of positive codimension[1].

By point 2, the zero set Z of the $(j^s\Lambda^*_r)^*\mathfrak{p}_i$ is non-singular close to e_0 and, by point 3, there is at least a hyperplane in $T_{e_0}E$ (actually, almost all of them) containing $T_{e_0}Z$ to which Λ_r is transversal. Equivalently, there is a linear combination $\lambda^i(j^s\Lambda^*_r)^*\mathfrak{p}_i = (j^s\Lambda^*_r)^*(\lambda^i\mathfrak{p}_i)$ such that $j^s\Lambda^*_r$ is transversal to $\mathfrak{p}_\lambda = \lambda^i\mathfrak{p}_i$ at e_0. By Hilbert's Nullstellensatz, since $\mathfrak{p}_\lambda$ clearly vanishes on Σ_0, there exist an integer exponent K such that $\mathfrak{p}^K_\lambda \in \mathfrak{P}_s$ and, correspondingly, an operator ${}_\lambda\mathcal{M}^*_s$, polynomial in $j^s\Lambda^*_r$, such that ${}_\lambda\mathcal{M}^*_s\mathcal{L}^*_r = \mathfrak{p}^K_\lambda\, i^0_{s+r}(G)$.

[1] See Appendix B about the transversality of PDOs.

Next, we use Lemma A.0.5 to find operators A_K and B_K of some order s' such that

$$\mathcal{L}_r A_K + p^K_\lambda B_K = i^0_{r+s'}(G)$$

and finally define $s'' = s + s'$ and

$$\mathcal{M}_{s''} = A_K +_\lambda \mathcal{M}_s B_K \,.$$

This $\mathcal{M}_{s''}$ is a right inverse for $\mathcal{L}_r$ since

$$\mathcal{L}_r \mathcal{M}_{s''} g = \mathcal{L}_r (A_K +_\lambda \mathcal{M}_s B_K) g = g - p^K_\lambda B_K g + \mathcal{L}_r \,{}_\lambda \mathcal{M}_s B_K g = g$$

for every $g \in \Gamma^{r+s''} G$. By Lemma A.0.5 the coefficients of $\mathcal{M}_{s''}$ are rational functions of $j^{s''}\Lambda^*_r$ which are regular at e_0. Then the polynomial $\mathfrak{q}$, defined as the product of all denominators of the coefficients of $\mathcal{M}_{s''}$, does not vanish at $j^{s''}\Lambda^*_r(e_0)$. On the other end $\mathfrak{q}\mathcal{M}^*_{s''}\mathcal{L}^*_r = \mathfrak{q}\, i^0_{s''+r}$ and $\mathfrak{q}\mathcal{M}^*_{s''}$ is polynomial in $j^s\Lambda^*_r$, so $\mathfrak{q} \in \mathfrak{P}_{s''}$. This $\mathfrak{q}$ is functionally independent on all polynomials in $\mathcal{P}_s$, or $(j^{s''}\Lambda^*_r)^*\mathfrak{q}$ would also vanish on e_0, so $\Sigma_{s''}$ has at least codimension $k+1$. It is true then that, given any set of $k \leq m$ functionally independent right inverses of $\mathcal{L}_r$, we can build a new one functioanlly independent on them, so that codim $\Sigma_s \geq m+1$. In particualr this means that the image $j^s\Lambda_r(E)$ of the s-jet of a generic linear under-determined PDO of order r does not intersects Σ_s, namely it admits a right inverse.

We can summarize all these results in the following statement:

Theorem 2.4.17 (Gromov, 1986). *Let F and G be vector bundles on E with $\dim E = m$, $\dim_E F = q$ and $\dim_E G = q'$. Then, if $q > q'$, for every r there exists a finite $s = s(q, q', n, r)$ such that a generic linear PDO*

$$\mathcal{L}_r : \Gamma^{r+s} F \to \Gamma^s G$$

is surjective. In particular, for every m, r and $q > q'$, we have that

$$\mathcal{L}_r(\Gamma^\infty F) = \Gamma^\infty G \,.$$

for a generic $\mathcal{L}_r$.

2.5 Non-free maps and the Gromov Conjecture

In Section 2.3.8(E') of [1] Gromov discusses the properties of isometric operators $\mathcal{D}_{M,q}$ in the cases $q < 2m + s_m$, when free maps are not dense, and $s_m < q < 2m + s_m$, when free maps cannot arise. In particular he poses the following question:

Question 2.5.1 (Gromov, 1986). *Are the operators $\mathcal{D}_{M,q}$ infinitesimally invertible over an open non-empty set for every $q > s_m$?*

Afterwards he conjectures that the theory of under-determined PDOs can be used to prove that the isometric operators $\mathcal{D}_{M,q}$ be infinitesimally invertible over a dense open set even when q is such that the set of free maps is not dense anymore and even when no free maps can arise:

Conjecture 2.5.2 (Gromov, 1986). *The operators $\mathcal{D}_{M,q}$ are infinitesimally invertible over a dense set for $q \geq m + s_m - \sqrt{m/2}$.*

In first subsection we show how Theorem 1.3.1 implies directly that the answer to question 2.5.1 is positive in the particular case $q = m + s_m - 1$, $M = \mathbb{R}^m$. In the second subsection we use the argument of Section 2.4 to make some step towards the proof of the general case.

2.5.1 $\mathcal{D}_{\mathbb{R}^m,q}$ is an open map over a non-empty open set for $q > m + s_m - 1$

Denote by $\mathcal{D}_{m,q}$ the operator $\mathcal{D}_{\mathbb{R}^m,q}$ acting on $C^\infty(\mathbb{R}^m, \mathbb{R}^q)$. It is well-known that $F(\mathbb{R}^m, \mathbb{R}^q)$ is non-empty for $q \geq q_m$ (see Theorem 2.3.5 and Example 2.3.6) so that, in particular, it turns out that there is a non-empty open set $\mathcal{A}$ on which the restriction of $\mathcal{D}_{m,q}$ is an open map for every $q \geq q_m$.

In a recent work by G. D'Ambra and A. Loi [25] steps were taken towards the proof of Conjecture 2.5.2 by showing, through an explicit construction that made use of the Lie equations after Gromov's idea in [1], p. 152, that $\mathcal{D}_{2,4}$ is open over a non-empty open set $\mathcal{A}^{DL}_{2,4}$. In this section we improve this result by extending it from $\mathcal{D}_{2,4}$ to all the $\mathcal{D}_{m,q}$ such that $q = q_m - 1$; as a byproduct, we also exhibit a larger set $\mathcal{A}_{2,4} \subset C^\infty(\mathbb{R}^2, \mathbb{R}^4)$ over which $\mathcal{D}_{2,4}$ is open.

Our argument is essentially based on Theorem 1.3.1 by Duistermaat and Hormander [20]. Let $q = q_m - 1$. Our aim is finding an open set $\mathcal{A}_{m,q} \subset C^\infty(\mathbb{R}^m, \mathbb{R}^q)$ such that, if $f_0 \in \mathcal{A}_{m,q}$ and $g_0 = \mathcal{D}_{m,q}(f_0)$, the equation

$$\mathcal{D}_{m,q}(f) = g \tag{2.16}$$

has solutions for every g close enough to g_0.

Recall that, by the Newton-Nash-Moser-Gromov IFT (Theorem 2.2.8), the existence of solutions of (2.16) is granted by the existence of solutions of its linearized version

$$2\delta_{ij}\partial_\alpha f^i \partial_\beta \delta f^j = \delta g_{\alpha\beta}$$

Following Gromov (see [1], Section 2.3.8 (E')) we set

$$\delta_{ij}\partial_\alpha f^i \delta f^j = h_\alpha$$

so that we get the following equivalent fully algebraic system:

$$\begin{cases} \delta_{ij}\ \partial_\alpha f^i \delta f^j & = h_\alpha \\ \delta_{ij}\ \partial_{\alpha\beta} f^i\, \delta f^j & = (\partial_\alpha h_\beta + \partial_\beta h_\alpha - \delta g_{\alpha\beta})/2 \end{cases} \tag{2.17}$$

where the h_α are m auxiliary arbitrary functions. Hence it is enough for our purposes to show that, for some non-empty open set of smooth functions, we can always choose the h_α so that system (2.17) has a solution.

Theorem 2.5.3. *If $q \geq q_m - 1$ there exist non empty open sets $\mathcal{A}_{m,q}$ such that the maps $\mathcal{D}_{m,q}\big|_{\mathcal{A}_{m,q}}$ are open.*

Proof. When $q \geq q_m$ the statement is trivially true because it is enough to choose $\mathcal{A}_{m,q} = F(\mathbb{R}^m, \mathbb{R}^q)$. We will assume therefore in the remainder of the proof that $q = q_m - 1$, i.e. that the number of equations is exactly one more than the number of unknowns δf^i.

Since the coefficients of the system (2.17) are exactly the components of the q_m vector fields $\{\partial_\alpha f, \partial_{\alpha\beta} f\}$, then clearly there exist non-identically zero functions λ^α and $\lambda^{\alpha\beta} = \lambda^{\beta\alpha}$ such that, identically,

$$\lambda^\alpha \partial_\alpha f + \lambda^{\alpha\beta} \partial_{\alpha\beta} f = 0.$$

This reflects in the following compatibility condition for system (2.17):

$$2\lambda^\alpha h_\alpha + \lambda^{\alpha\beta}(\partial_\alpha h_\beta + \partial_\beta h_\alpha - \delta g_{\alpha\beta}) = 0\,.$$

It is convenient to rewrite this as the cohomological equation

$$X^\alpha h_\alpha = \phi\,, \tag{2.18}$$

where $\phi = \lambda^{\alpha\beta}\delta g_{\alpha\beta}$, X^α is the first-order non-homogeneous differential operator $X^\alpha = L_{\xi_\alpha} + 2\lambda^\alpha$, L_{ξ_α} is the Lie derivative with respect to the vector field $\xi_\alpha = \lambda^{\alpha\beta}\partial_\beta$ and the smooth functions λ^α must be thought as multiplication operators.

Now, let $\mathcal{A}_{m,q} \subset C^\infty(\mathbb{R}^m, \mathbb{R}^q)$ be the open set of immersions f satisfying the following two open properties: 1. the $q_m \times (q_m - 1)$ matrix $D^2 f$ has full-rank at every point; 2. there is an index α_0 such that the functions $\lambda^{\alpha_0\beta}$ are never all zero at the same time. Then, after setting $h_\beta = \lambda^{\alpha_0\beta} h$, $\beta = 1, \ldots, m$, for some unknown function h, equation (2.18) becomes

$$Yh = \phi'$$

where $Y = L_\zeta + \lambda'$ for some vector field ζ and function λ'. A direct computation shows that the component α_0 of ζ is equal to $(\lambda^{\alpha_0 1})^2 + \cdots + (\lambda^{\alpha_0 m})^2$ and therefore it is never zero by hypothesis. In particular this means that every surface $x^{\alpha_0} = const$ is a global transversal for ζ and therefore, by Theorem DH, Y is a surjective first-order partial differential operator. Hence for every function belonging to $\mathcal{A}_{m,q}$ it is always possible to choose the h_α in function of the $\delta g_{\alpha\beta}$ so that the compatibility condition (2.18) is satisfied. Examples 1 and 2 show that these sets are non-empty. □

Example 2.5.4. *Consider any pair (g, h) of free maps from $\mathbb{R}$ to $\mathbb{R}^2$. Then the function $f_{gh} : \mathbb{R}^2 \to \mathbb{R}^4$ defined by $f_{gh}(x, y) = (g(x), h(y))$ belongs to $\mathcal{A}_{2,4} \subset C^\infty(\mathbb{R}^2, \mathbb{R}^4)$. Indeed in this case $\partial_{xy} f_{gh} = 0$, so that we can choose*

$$\lambda^x = \lambda^y = \lambda^{xx} = \lambda^{yy} = 0\,,\ \lambda^{xy} = \lambda^{yx} = 1$$

and therefore the compatibility condition becomes simply

$$\partial_x h_y + \partial_y h_x = \delta g_{xy}$$

which is trivially solvable. E.g. the function $f(x,y) = (x, e^x, y, e^y)$ belongs to $\mathcal{A}_{2,4}$. Note that, while one can easily check that every function of $\mathcal{A}^{DL}_{2,4}$ also belongs to $\mathcal{A}_{2,4}$, the function f does not belong to $\mathcal{A}^{DL}_{2,4}$, i.e. $\mathcal{A}_{2,4}$ is strictly larger of the set introduced in [25].

Remark 2.5.5. *Let $\mathcal{D}_q$ be the metric-inducing operator acting on $C^\infty(\mathbb{T}^2, \mathbb{R}^q)$. As mentioned in the introduction, the cases $q \geq 7$ and $q < 4$ are trivial. Only for $q = 4$, among the non-trivial cases, free maps cannot arise but Example 1 can be used to show that $\mathcal{D}_4$ is, nevertheless, open over a non-empty open set. Indeed the set $\mathcal{A}_{2,4}$ contains functions periodic in both x and y, e.g. $f(x,y) = (\cos x, \sin x, \cos y, \sin y)$. The subset of all of them, considered as functions on $\mathbb{T}^2$, is open in $C^\infty(\mathbb{T}^2, \mathbb{R}^4)$ and $\mathcal{D}_4$ is clearly an open map over it.*

Example 2.5.6. *Let $f \in F(\mathbb{R}^m, \mathbb{R}^{q_m})$ be the canonical free map given by*

$$f(x^1, \dots, x^m) = (x^1, \dots, x^m, (x^1)^2, x^1x^2, \cdots, (x^m)^2)$$

and π any projection $\pi : \mathbb{R}^{q_m} \to \mathbb{R}^{q_m - 1}$ which "forgets" any one of the last $(m+1)/2$ components. Then the composition $f_\pi = \pi \circ f$ belongs to $\mathcal{A}_{m,q_m-1}$. Indeed the matrix $D^2 f_\pi$ has full rank and one of the second derivatives of f_π (say $\partial_{x^1x^2} f_\pi$) is identically zero, so we can choose the corresponding factor ($\lambda^{x^1x^2}$ in this case) identically equal to 1 and all others equal to zero. For example, in the $(m,q) = (2,4)$ case we get the functions $f_1(x,y) = (x, y, xy, y^2)$, $f_2(x,y) = (x, y, x^2, y^2)$ and $f_3(x,y) = (x, y, x^2, xy)$.

Note that, exactly like in [25], the set of $q_m \times (q_m - 1)$ matrices not satisfying the conditions that define the open sets $\mathcal{A}_{m,q}$ has just codimension 1 in the fibers of the bundle $J^2(\mathbb{R}^m, \mathbb{R}^q) \to J^0(\mathbb{R}^m, \mathbb{R}^q)$ while we would need at least codimension 3 in order to apply the transversality theorems. In particular the sets $\mathcal{A}_{m,q}$ are not dense in $C^\infty(\mathbb{R}^m, \mathbb{R}^q)$.

2.5.2 Infinitesimal invertibility of $\mathcal{D}_{M,q}$ on non-free isometric immersions for $q > s_m$

Now consider the general case and recall (see Example 2.2.6 and Example 2.4.16) that the linearization of $\mathcal{D}_{M,q}$ is given by

$$\ell_{M,q}(f, \delta f) = 2\delta_{ij} \partial_\alpha f^i \partial_\beta \delta f^j \, dx^\alpha \otimes dx^\beta$$

whose adjoint is

$$\ell^*_{M,q}(f, \delta g) = 2\, \partial^{\alpha\beta} f^i \, \delta g_{\alpha\beta} \, \partial_i - (\delta^{\gamma\alpha} \, \partial^\beta f^i + \delta^{\gamma\beta} \, \partial^\alpha f^i) \partial_\gamma \delta g_{\alpha\beta} \, \partial_i$$

where we set $\partial^\alpha = \delta^{\alpha\alpha'} \partial_{\alpha'}$ and similarly for the second derivatives in order to use the Einstein summation convention.

In order to make notations as easy to read as possible, we denote the coordinates in the fibers of $S^0_s M$, i.e. a choice of independent components of $\delta_{\alpha\beta}$, by pairs $(\alpha\beta)$, $\alpha \leq \beta$. Then the homomorphism $\Lambda^*_{M,q} : J^1(S^0_2 M) \to J^0(M, \mathbb{R}^q)$ has components

$$\begin{cases} \Lambda^i_{(\alpha\beta)} &= 2\,\partial^{\alpha\beta} f^i \\ \Lambda^{i\gamma}_{(\alpha\beta)} &= -(\delta^\gamma_\alpha\,\partial_\beta f^i + \delta^\gamma_\beta\,\partial_\alpha f^i)\,. \end{cases}$$

With these notations system (2.13), whose coefficients are the fibers coordinates on $J^s \operatorname{Hom}(J^1(S^0_2 M), J^0(M, \mathbb{R}^q))$, writes as

$$\begin{cases} \sum\limits_{|A|\leq s} M_i^{(\alpha_0\beta_0)A} \Lambda^i_{(\alpha\beta)A} = \delta^{(\alpha_0\beta_0)}_{(\alpha\beta)} \\ \sum\limits_{|A|\leq s} M_i^{(\alpha_0\beta_0)A} \Lambda^{i\beta_1}_{(\alpha\beta)A} + \sum\limits_{|A|\leq s-1} M_i^{(\alpha_0\beta_0)\beta_1 A} \Lambda^i_{(\alpha\beta)A} = 0 \\ \sum\limits_{|A|\leq s-1} M_i^{(\alpha_0\beta_0)\beta_1 A} \Lambda^{i\beta_2}_{(\alpha\beta)A} + \sum\limits_{|A|\leq s-2} M_i^{(\alpha_0\beta_0)\beta_1\beta_2 A} \Lambda^i_{(\alpha\beta)A} = 0 \\ \vdots \\ \sum\limits_{|A|\leq 1} M_i^{(\alpha_0\beta_0)\beta_1\ldots\beta_{s-1} A} \Lambda^{i\beta_s}_{(\alpha\beta)A} + M_i^{(\alpha_0\beta_0)\beta_1\ldots\beta_s} \Lambda^i_{(\alpha\beta)} = 0 \\ M_i^{(\alpha_0\beta_0)\beta_1\ldots\beta_s} \Lambda^{i\beta_{s+1}}_{(\alpha\beta)} = 0 \end{cases} \tag{2.19}$$

Note that $\ell_{M,q}$ is quite far from being generic, since only mq of its mqs_m components $(\Lambda_{M,q})^\gamma_{(\alpha\beta)i}$ are independent. Thus, we cannot apply Theorem 2.4.17 to it. Nevertheless observe that, in the non-homogeneous row, only the s-jets of the zero-order components $\Lambda^i_{(\alpha_0\beta_0)} = 2f^i_{\alpha_0\beta_0}$ appear and the $f^i_{\alpha_0\beta_0}$ are, on the contrary, all independent.

Below we follow closely Gromov's argument used in the proof of Theorem 2.4.17. System (2.19) admits a solution iff the non-homogeneous row

$$2 \sum_{|A|\leq s} M_i^{(\alpha_0\beta_0)A} f^i_{\alpha_0\beta_0 A} = 1$$

is not a linear combination, with coefficients $\lambda_k \in \mathcal{R}$, of the remaining (homogeneous) rows, where $\mathcal{R}$ is the ring of rational functions in the coordinates of the fibers of $J^s \operatorname{Hom}(J^1(S^0_2 M), J^0(M, \mathbb{R}^q))$.

Assume that such combination exists. We have to treat differently the case when α_0 and β_0 are equal and the one when they are different.

Case 1, $\alpha_0 = \beta_0$. We take, for the argument's sake, $(\alpha_0\beta_0) = (1,1)$. First of all we observe that the column of the unknown $M_i^{(11)}$ has f^i_{11} as coefficient in the non-homogeneous row and the f^i_α and $f^i_{\alpha\beta}$, with $(\alpha\beta) \neq (1,1)$, in all other rows. Since none of these appear elsewhere in the non-homogeneous row, we express f^i_{11} as linear combination of these functions and substitute this expression in the rest of the system.

Next, we look at the columns corresponding to the unknowns $M_i^{(11)\alpha_1}$. Consider first the terms with $\alpha_1 \neq 1$. Their coefficient is $f^i_{11\alpha_1}$. In the same column appear all other coefficients $f^i_{\alpha\beta\alpha_1}$, with $(\alpha\beta) \neq (1,1)$, and the coefficients

$\Lambda^{i\beta_1}_{(\alpha\beta)\alpha_1}$, which contain only second derivatives of the f^i. Hence we express all the $f^i_{11\alpha_1}$, $\alpha_1 \neq 1$, in terms of all other third order derivatives of f^i not having two indices equal to 1 and lower order derivatives and substitute this expression in the rest of the system.

Now consider the column of $M_i^{(11)1}$, whose coefficient is f^i_{111}. Besides terms of lower order, this column contains all other terms of the kind $f^i_{\alpha\beta 1}$, with $(\alpha\beta) \neq (1,1)$. Among these terms there are all the $f^i_{11\alpha_1}$, α_1, which we already expressed in terms of the other third derivatives of the f^i, so that also f^i_{111} ends up expressed in terms of the same coefficients of the $f^i_{11\alpha_1}$, α_1.

Finally let us discuss the general case of the columns corresponding to the unknowns $M_i^{(11)\alpha_1\ldots\alpha_k}$. Following what we just done, we start from the ones such that $\alpha_i \neq 1$, $i = 1,\ldots,m$. No other coefficient $\Lambda^i_{(\alpha\beta)\alpha_1\ldots\alpha_k}$, with $(\alpha\beta) \neq (1,1)$, (i.e. a derivative of f^i of order $k+2$) in the column of $M_i^{(11)\alpha_1\ldots\alpha_k}$ appears as coefficient in other positions in the non-homogeneous row, since these coefficients have at most one lower index 1 (either α or β) while those in the non-homogeneous row have at least two of them. Hence we can write all the coefficients $f_{(11)\alpha_1\ldots\alpha_k}$ in terms of coefficients of the same order not appearing in the non-homogeneous row and coefficients of lower order. We replace all these expressions in the rest of the system and now consider the case when $\alpha_1 = 1$ and $\alpha_i \neq 1$, $i = 2,\ldots,m$. The only coefficients belonging to the non-homogeneous row that could appear in the same columns corresponding to these terms are the ones with $\alpha_i \neq 1$, $i = 1,\ldots,m$, which we just replaced, since each of such terms will have at most two lower indices equal to 1. Hence also these terms are now expressed in terms of derivatives of f^i which do not appear in the non-homogeneous row. Operating recursively we end up expressing all terms of order k as functions of the λ_k and of derivatives of f^i up to order k none of which appears in the non-homogeneous row.

By a standard induction argument it is then clear that, assuming that the non-homogeneous row is a linear combination of the homogeneous ones, we can express all coordinates f^i_{11A}, $|A| \leq s$, in terms of the remaining coordinates (up to order $s+2$) and of the coefficients λ_k.

Case 2, $\alpha_0 \neq \beta_0$. We take, for the argument's sake, $(\alpha_0\beta_0) = (12)$. As in case 1, in the homogeneous rows of the column of $M_i^{(12)}$ appear all $f^i_{(\alpha\beta)}$, $\{\alpha,\beta\} \neq \{1,2\}$, none of which appears in the non-homogeneous row. Hence we can express $f^i_{(12)}$ through those coefficients and substitute its expression anywhere else in the system.

Let us consider now the column of $M_i^{(12)\alpha_1}$. If $\alpha_1 \notin \{1,2\}$ then no coefficient $f^i_{\alpha\beta\alpha_1}$ in the same column appears in the non-homogeneous row since they lack either a 1 or a 2 among their lower index. All other coefficients appearing in the same columns are of lower order, so these $f^i_{\alpha\beta\alpha_1}$ can be expressed in terms of coefficients not appearing anywhere in the non-homogeneous row. As usual, we substitute their expression anywhere else in the system. When instead $\alpha_1 \in \{1,2\}$, then it is easy to realize that f^i_{121} appears in the same column of f^i_{122} and viceversa, while all other coefficients do not appear anywhere in the non-homogeneous row. In this case we have then the following situation:

$$f^i_{121} = \mu f^i_{122} + \sum\nolimits^h \mu_i^{(\alpha\beta)A} f^i_{\alpha\beta A}\,,\ f^i_{122} = \nu f^i_{121} + \sum\nolimits^h \nu_i^{(\alpha\beta)A} f^i_{\alpha\beta A}$$

where the coefficients $\mu, \mu_i^{(\alpha\beta)A}, \nu, \nu_i^{(\alpha\beta)A}$ are polynomials in $\mathcal{R}$ and the sum $\sum^h$ is extended only to the coefficients that *do not* appear in the non-homogeneous row. Then

$$f^i_{122} = \nu\mu f^i_{122} + \sum\nolimits^h (\nu')_i^{(\alpha\beta)A} f^i_{\alpha\beta A}$$

so that either $\nu\mu = 1$ or

$$f^i_{122} = (1-\nu\mu)^{-1}\left(\sum\nolimits^h (\nu')_i^{(\alpha\beta)A} f^i_{\alpha\beta A}\right).$$

If $\nu\mu = 1$ we cannot say anything on f^i_{122}, so that we must decrease by 1 the count of the coefficients f^i_{12A}, but we found out that there is a relation between the coefficients λ_k, so we must decrease by one also the number of (independent) coefficients λ_k. Hence, for the sake of the argument, we can safely assume that $\nu\mu \neq 1$. In this case, f^i_{122} can be written as linear combination of coefficients f^i_{12A} of equal or lower order that *do not* appear in the non-homogeneous row. The coefficients of this linear combination are *rational* functions of the original λ_k, so they still belong to $\mathcal{R}$. Finally we substitute back this expression in the one for f^i_{122} so that are able to express all coefficients $f^i_{12\alpha_1}$, $\alpha_1 = 1, \dots, m$, as linear combinations, with coefficients in $\mathcal{R}$, of coefficients $f^i_{\alpha\beta A}$ of equal or lower order which do not appear in the non-homogeneous row.

Now consider the general case of terms of order n starting from $M_i^{(12)\alpha_1\dots\alpha_n}$ with $\{\alpha_1, \dots, \alpha_n\} \cap \{1,2\} = \emptyset$, whose coefficient is $\Lambda^i_{(12)\alpha_1\dots\alpha_n} = f^i_{(12)\alpha_1\dots\alpha_n}$. Clearly in the column of such term no other same-order coefficient $f^i_{(\alpha\beta)\alpha_1\dots\alpha_n}$, with $\{\alpha,\beta\} \neq \{1,2\}$, appears in the non-homogeneous one since they all lack either a 1 or a 2 among the lower index. As usual, we substitute the expressions of the $M^{(12)\alpha_1\dots\alpha_n}$ in the system and then consider the case of the columns $M^{(12)\alpha_1\dots\alpha_n}$ where either $\alpha_1 = 1$ or $\alpha_1 = 2$ and $\{\alpha_2, \dots, \alpha_n\} \cap \{1,2\} = \emptyset$. The very same considerations made above for the case of $M_i^{(12)\alpha_1}$ take care of this case. Now consider the case of the columns of $M^{(12)\alpha_1\dots\alpha_n}$ with either $\alpha_1 = 1$ or $\alpha_1 = 2$ *and* either $\alpha_2 = 1$ or $\alpha_2 = 2$. In this case we are in the following situation:

$$\begin{cases} f^i_{1211\alpha_3\dots\alpha_n} &= \mu f^i_{2211\alpha_3\dots\alpha_n} + \sum^h \mu_i^{(\alpha\beta)A} f^i_{(\alpha\beta)A} \\ f^i_{1212\alpha_3\dots\alpha_n} &= \nu f^i_{1211\alpha_3\dots\alpha_n} + \nu' f^i_{2212\alpha_3\dots\alpha_n} + \sum^h \nu_i^{(\alpha\beta)A} f^i_{(\alpha\beta)A} \\ f^i_{1222\alpha_3\dots\alpha_n} &= \phi f^i_{1122\alpha_3\dots\alpha_n} + \sum^h \phi_i^{(\alpha\beta)A} f^i_{(\alpha\beta)A}\,, \end{cases}$$

where, as above, all functions denoted with greek letters are polynomials in $\mathcal{R}$ and $\sum^h$ is extended only to terms which do not appear in the non-homogeneous row. We substitute the expressions of $f^i_{1211\alpha_3\dots\alpha_n}$ and $f^i_{1222\alpha_3\dots\alpha_n}$ in $f^i_{1212\alpha_3\dots\alpha_n}$ and repeat the argument above: if $\nu\mu + \nu'\phi = 1$ then we cannot say anything on $f^i_{1212\alpha_3\ldots\alpha_n}$ but we have a relation among the λ_k so the balance between the number of coefficients in the non-homogeneous row and the number of λ_k does not change. Hence we assume, for the argument's sake, that $\nu\mu + \nu'\phi \neq 1$ and replace its expression, now as function only of coefficients which do not appear anywhere in the non-homogeneous row, back in $f^i_{1211\alpha_3\ldots\alpha_n}$ and $f^i_{1222\alpha_3\ldots\alpha_n}$. Thus all coefficients $f^i_{12\alpha_1\ldots\alpha_n}$ with either $\alpha_1 = 1$ or $\alpha_1 = 2$ and either $\alpha_2 = 1$ or $\alpha_2 = 2$ can be expressed as linear combinations of similar terms of the same or lower order that do not appear in the homogeneous rows. We replace all of them elsewhere in the system and continue.

When we consider the case when more α_i can be equal to 1 or 2 the situation does not change qualitatively. We have some finite number of $f^i_{(12)A}$ with the following property: two of them (the "first" and the "last" contain in their column another term appearing in the non-homogeneous row (the "second" and the "next-to-last"); the "second" has to of them, the "first" and the "third" and so on until the "next-to-last" and the "last". Following the steps above we can express each one of them as a linear combination of coefficients $f^i_{(12)A}$ of same or lower order which do not appear in the non-homogeneous row. The coefficients of this linear combination are all rational functions of the λ_k and therefore also belong to $\mathcal{R}$.

It is clear then that it is possible to repeat this procedure until all of the $f^i_{(12)A}$ in the non-homogeneous row are expressed as linear combination of the $f^i_{(12)A}$ *not* appearing in the non-homogeneous row.

The arguments above shows that in the system corresponding to the index $(\alpha_0\beta_0)$, whether we are in case 1 or case 2, the number of the (independent) $f^i_{(\alpha\beta)A}$ in the non-homogeneous row is $q\binom{m+s}{s}$. The number of homogeneous rows of system (2.19) is $s_m\binom{m+s+1}{s+1}$. Clearly, if a linear combination of the homogeneous rows is equal to the non-homogeneous row, the $f^i_{(\alpha\beta)A}$ must be contained in the λ_k and so there cannot be fewer rows than coefficients $f^i_{(\alpha\beta)A}$. As discussed in the proof of Theorem 2.4.17, if $q \geq s_m$ and

$$s > \frac{m s_m}{q - s_m}$$

then there are more $f^i_{(\alpha\beta)A}$ than rows and therefore $l^*_{M,q}$ admits a *formal* left inverse.

This inverse is regular outside of the set of zeros of the coefficients of the inverse. The set of partial differential inequalities obtained by setting all denominators different from zero defines an open subset $\mathcal{A}_q$ of $C^\infty(M, \mathbb{R}^q)$. In order to solve the problem of Gromov now it is needed to study when $\mathcal{A}_q$ is non-empty.

3

$\mathcal{H}$-Free Maps and infinitesimal invertibility of the $\mathcal{H}$-isometric operator

In this chapter we extend the theory of isometric embeddings of a manifold M into $\mathbb{R}^q$ by considering maps $M \to \mathbb{R}^q$ which are injective on some fixed distribution $\mathcal{H} \subset TM$.

In Section 3.1 we define the concept of $\mathcal{H}$-free map, which reduces to the one of free map for $\mathcal{H} = TM$, and of the $\mathcal{H}$-isometric operator $\mathcal{D}_{\mathcal{H}}$. In this setting we prove Theorem 3.1.11 about the existence and density of $\mathcal{H}$-free maps, which is the analog of Theorem 2.3.2 and Proposition 2.3.4 for free maps, and Theorem 3.1.13, which is the analog of Nash's Theorem 2.3.3, showing that $\mathcal{D}_{\mathcal{H}}$ is an open map over the set of $\mathcal{H}$-free maps.

After showing several concrete examples of $\mathcal{H}$-free maps for distributions of dimension or codimension equal to 1 (Section 3.1.1), in Section 3.2 we prove the existence of $\mathcal{H}$-free maps in critical dimension in the following cases: one-dimensional distributions of finite-type on $\mathbb{R}^2$; Lagrangian distributions of Complete Integrable Systems; Hamiltonian distributions of Riemann-Poisson brackets.

The contents of this chapter have been published, as a joint work with G. D'Ambra and A. Loi, in [3].

3.1 $\mathcal{H}$-free maps and the linearization of the operator $\mathcal{D}_{\mathcal{H}}$

Let $\mathcal{H}$ be a k-dimensional distribution on M, i.e. a vector subbundle of TM. Fix local coordinates (x^α) on some chart $U \subset M$, $\alpha = 1, \cdots, m$, and let $\{\xi_a\}$, $a = 1, \cdots, k$, be a local trivialization for $\mathcal{H}$ in U, so that $\mathcal{H}|_U = \text{span}\{\xi_1, \ldots, \xi_k\}$. Let $\{\theta^a, \omega^A\}$, $A = 1, \cdots, m-k$, be a dual base for the whole T^*M such that

$i_{\xi_b}\theta^a = \delta^a_b$ and $i_{\xi_b}\omega^A = 0$. Then

$$\mathcal{H}|_U = \bigcap_{A=1}^{m-k} \ker \omega^A.$$

and the gradient of the components of a C^r map $f = (f^1, \dots, f^q) : M \to \mathbb{R}^q$ writes

$$df^i = u^i_A \omega^A \oplus v^i_a \theta^a \ , \ \ i = 1, \cdots, q$$

where $v^i_a = i_{\xi_a} df^i = L_{\xi_a} f^i$ (the u^i_A play no role in what follows). Then, in local terms, the restriction to $\mathcal{H} \subset TM$ of $f^* e_q = \delta_{ij} df^i \otimes df^j$ is given by

$$f^* e_q|_{\mathcal{H}} = \delta_{ij} L_{\xi_a} f^i L_{\xi_b} f^j \ \theta^a \otimes \theta^b \in \Gamma^0(S^0_2 \mathcal{H}) \tag{3.1}$$

and the equation $\mathcal{D}_{\mathcal{H}}(f) = g$ writes locally as

$$\delta_{ij} L_{\xi_a} f^i L_{\xi_b} f^j = g_{ab}, \tag{3.2}$$

where $g_{ab} = g(\xi_a, \xi_b)$, $a, b = 1, \dots, k$.

Definition 3.1.1. *Let $\mathcal{H}$ be a distribution on M. We say that $f \in C^1(M, \mathbb{R}^q)$ is an* $\mathcal{H}$-immersion *if the restriction of Tf to $\mathcal{H}$ is injective.*

Example 3.1.2. *Take $M = A \times B$, where A and B are smooth manifolds. Consider the two natural projections π_A and π_B on A and B and the corresponding two canonical distributions $\mathcal{H}_A = \ker T\pi_B = TA \oplus \{0\}$ and $\mathcal{H}_B = \ker T\pi_A = \{0\} \oplus TB$. A map $f \in C^\infty(M, \mathbb{R}^q)$ is a $\mathcal{H}_A$-immersion iff $f(\cdot, b) : A \to \mathbb{R}^q$ is an immersion for every $b \in B$. Similarly for $\mathcal{H}_B$.*

Example 3.1.3. *For any fiber bundle (M, N, π, F) it is defined the canonical distribution of vertical vectors $V = \ker T\pi \subset TM$. A smooth map $f : M \to \mathbb{R}^q$ is a V-immersion iff on every trivialization $U \times F$ of M the map $f(u, \cdot) : F \to \mathbb{R}^q$ is an immersion for every $u \in U$. Let now A be a linear connection on M and let H be the horizontal distribution with respect to A; then a map $f : M \to \mathbb{R}^q$ is a H-immersion iff the covariant derivatives $\{\nabla_\mu f^i\}$ are linearly independent on every point of M.*

Proposition 3.1.4. *Let $f \in C^r(M, \mathbb{R}^q)$. The quadratic form $\mathcal{D}_{\mathcal{H}}(f) \in \Gamma^0(S^0_2 \mathcal{H})$ is positive-definite iff f is a $\mathcal{H}$-immersion.*

Proof. Let $\{\xi_a\}$ be a local trivialization for $\mathcal{H}$. Then

$$Tf(\xi_a) = \partial_\alpha f^i \partial_i \otimes dx^\alpha(\xi^\beta_a \partial_\beta) = \xi^\alpha_a \partial_\alpha f^i \partial_i = (L_{\xi_a} f^i)\partial_i$$

and the proposition follows. □

Proposition 3.1.5. *Let $\mathcal{H} \subset TM$ be a k-dimensional distribution. If $q \geq m+k$ the set of $\mathcal{H}$-immersions is open and dense in $C^1(M, \mathbb{R}^q)$.*

Proof. A map $f : M \to \mathbb{R}^q$ is a $\mathcal{H}$-immersion iff the $k \times q$ matrix $D = (L_{\xi_a} f^i) : M \to M_{k,q}(\mathbb{R})$ has rank k at every point. The set of non maximal rank matrices has codimension $q - k + 1$ in $M_{k,q}(\mathbb{R})$ [24] so the image $D(M)$ do not intersect it if $m < q - k + 1$. □

Let us consider now a smooth 1-parameter deformation g_ϵ of the metric g on M such that $g_0 = g$ and assume that there exists a corresponding smooth 1-parameter deformation f_ϵ such that $f_0 = f$. It follows by (3.2) that

$$\delta_{ij} L_{\xi_a} f_\epsilon^{\,i} L_{\xi_b} f_\epsilon^{\,j} = g_{\epsilon,ab} \ ,$$

where $g_{\epsilon,ab} = g_\epsilon(\xi_a, \xi_b)$. Differentiate with respect to ϵ and set

$$\delta f^i = \left.\frac{df^i_\epsilon}{d\epsilon}\right|_{\epsilon=0} , \quad \delta g_{ab} = \left.\frac{dg_{\epsilon,ab}}{d\epsilon}\right|_{\epsilon=0}$$

thus obtaining the system of $k(k+1)/2$ PDEs:

$$\delta_{ij}\big(L_{\xi_a} f^i(x)\delta[L_{\xi_b} f^j(x)] + \delta[L_{\xi_a} f^i(x)]L_{\xi_b} f^j(x)\big) = \delta g_{ab}(x).$$

Following Nash we observe that

$$L_{\xi_a} f^i \delta[L_{\xi_b} f^j] = L_{\xi_b}[L_{\xi_a} f^i \delta f^j] - L_{\xi_b} L_{\xi_a} f^i \delta f^j$$

so that, by defining $\psi_a(x) = \delta_{ij} L_{\xi_a} f^i(x) \delta f^j(x)$, we get the following equivalent algebraic system in the q unknown δf^j:

$$\begin{cases} \delta_{ij} \ L_{\xi_a} f^i \delta f^j & = \psi_a \\ \delta_{ij} \ (L_{\xi_a} L_{\xi_b} f^i + L_{\xi_b} L_{\xi_a} f^i)\delta f^j & = L_{\xi_a}\psi_b + L_{\xi_b}\psi_a - \delta g_{ab} \end{cases} \tag{3.3}$$

where the ψ_a are arbitrary functions.

A sufficient condition for this system to be solvable is that the matrix

$$D_{\xi_1,\cdots,\xi_k,f} = \begin{pmatrix} L_{\xi_1} f^1 & \cdots & L_{\xi_1} f^q \\ \vdots & \vdots & \vdots \\ L_{\xi_k} f^1 & \cdots & L_{\xi_k} f^q \\ L^2_{\xi_1} f^1 & \cdots & L^2_{\xi_1} f^q \\ L_{\xi_1} L_{\xi_2} f^1 + L_{\xi_2} L_{\xi_1} f^1 & \cdots & L_{\xi_1} L_{\xi_2} f^q + L_{\xi_2} L_{\xi_1} f^q \\ \vdots & \vdots & \vdots \\ L^2_{\xi_k} f^1 & \cdots & L^2_{\xi_k} f^q \end{pmatrix} \tag{3.4}$$

has maximal rank, i.e. that the vectors

$$L_{\xi_a} f^i, \ \{L_{\xi_a}, L_{\xi_b}\} f = \ L_{\xi_a} L_{\xi_b} f^i + L_{\xi_b} L_{\xi_a} f^i$$

be linearly independent. Note that matrix (3.4) has always at least 2 rows so we must assume $q \geq 2$.

Definition 3.1.6. *Let $\mathcal{H} \subset TM$ be a k-dimensional distribution on a smooth manifold M. We say that a C^2 $\mathcal{H}$-immersion $f : M \to \mathbb{R}^q$ is* $\mathcal{H}$-free *if, for every $x \in M$, there exists a trivialization $\{\xi_a\}$ of $\mathcal{H}$ in some neighbourhood of x such that*

$$\operatorname{rank} D_{\xi_1,\cdots,\xi_k,f} = k + s_k,\ \ s_k = \frac{k(k+1)}{2}\ .$$

The set of C^r $\mathcal{H}$-free maps $M \to \mathbb{R}^q$ is denoted by $F^r_{\mathcal{H}}(M, \mathbb{R}^q)$.

Remark 3.1.7. *$\mathcal{H}$-free maps where defined first in [26] but used there for different purposes.*

Clearly $\mathcal{H}$-free maps can exist only for $q \geq k+s_k$. Note also that, while every immersion is a $\mathcal{H}$-immersion, it is not necessarily $\mathcal{H}$-free, e.g. for dimensional reasons. Next proposition shows that the above definition is well posed.

Proposition 3.1.8. *The rank of the matrix $D_{\xi_1,\cdots,\xi_k,f}$ does not depend on the particular choice of the trivialization of $\mathcal{H}$.*

Proof. Take another trivialization $\{\zeta_a\}$ of $\mathcal{H}$ in the same neighbourhood of x. Then $\zeta_a(x) = \lambda^b_a(x)\xi_b(x)$ for some local section $\lambda^b_a(x)$ of the frame bundle over $\mathcal{H}$ and

$$L_{\zeta_a} f = \lambda^b_a L_{\xi_b} f\ ,\ \ L_{\zeta_a} L_{\zeta_b} f = \lambda^c_a L_{\xi_c} \lambda^d_b L_{\xi_d} f + \lambda^c_a \lambda^d_b L_{\xi_c} L_{\xi_d} f\ .$$

Clearly $\operatorname{rank}(L_{\zeta_a} f^i) = \operatorname{rank}(L_{\xi_a} f^i)$. Hence

$$\operatorname{rank}\begin{pmatrix} L_{\zeta_a} f \\ \{L_{\zeta_a}, L_{\zeta_b}\} f \end{pmatrix} = \operatorname{rank}\begin{pmatrix} L_{\xi_a} f \\ (\lambda^c_a L_{\xi_c} \lambda^d_b + \lambda^c_b L_{\xi_c} \lambda^d_z) L_{\xi_d} f + \lambda^c_a \lambda^d_b \{L_{\xi_a}, L_{\xi_b}\} f \end{pmatrix}$$

$$= \operatorname{rank}\begin{pmatrix} L_{\xi_a} f \\ \lambda^c_a \lambda^d_b \{L_{\xi_a}, L_{\xi_b}\} f \end{pmatrix} = \operatorname{rank}\begin{pmatrix} L_{\xi_a} f \\ \{L_{\xi_a}, L_{\xi_b}\} f \end{pmatrix}$$

□

Example 3.1.9. *In [27] Kaplan showed that every one-dimensional distribution $\mathcal{H}$ on the plane $\mathbb{R}^2$ is orientable and then it is the kernel of a regular [1] 1-form ω. The metric induced on $\mathcal{H} = \ker \omega$ by a map $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (\alpha(x, y), \beta(x, y))$, is, by Eq. (3.2),*

$$\mathcal{D}_{\mathcal{H}}(f) = [(L_\xi \alpha)^2 + (L_\xi \beta)^2](*\omega)^2$$

where $*$ is the Euclidean Hodge operator. Then f is a $\mathcal{H}$-immersion iff $L_\xi \alpha$ and $L_\xi \beta$ do not vanish simultaneously at any point. Here the matrix $D_{\xi,f}$ is given by the 2×2 matrix

$$D_{\xi,f} = \begin{pmatrix} L_\xi \alpha & L_\xi \beta \\ L^2_\xi \alpha & L^2_\xi \beta \end{pmatrix}\ .$$

[1] Throughout this paper we call a vector field or a k-form *regular* if they are different from zero at every point of M.

Therefore f is $\mathcal{H}$-free iff there is a regular section ξ of $\mathcal{H}$ such that

$$L_\xi \alpha \; L^2_\xi \beta - L_\xi \beta \; L^2_\xi \alpha > 0$$

on the whole plane $\mathbb{R}^2$.

The following proposition, analogue of Theorem A in Appendix 6 of [12], characterizes $\mathcal{H}$-free maps:

Proposition 3.1.10. *Let $\mathcal{H}$ be a distribution on a smooth manifold M, let $S = S^0_2\mathcal{H}$ be the set of its symmetric $(0,2)$ tensors on $\mathcal{H}$ and $N = (Tf(\mathcal{H}))^\perp$ the normal bundle to $Tf(\mathcal{H})$ in $\mathbb{R}^q$ with respect to e_q. Then a $\mathcal{H}$-immersion $f : M \to \mathbb{R}^q$ is $\mathcal{H}$-free iff the "Wintergarten map" $\nu : N \to S$ defined by*

$$\nu_x(n_x) = \{L_{\xi_a}, L_{\xi_b}\} f^i(x) \delta_{ij} n^j_x \, \theta^a \otimes \theta^b$$

is surjective.

Proof. Assume first that f is $\mathcal{H}$-free. Then $\ker \nu$ cannot be bigger than the zero-section of N because the existence of a non-zero vector $n_x \in N_x$ such that $\nu_x(n_x) = 0$ is equivalent to the existence of a non-trivial linear relation between the vectors $l_{a,b} = (\{L_{\xi_a}, L_{\xi_b}\} f^i)$, which are linearly independent by the $\mathcal{H}$-freedom of f.

Viceversa assume that ν is surjective. Note that the matrix representing ν as a linear operator is exactly the $s_k \times q$ matrix $D_{\xi_1,\cdots,\xi_k,f} = (\{L_{\xi_a}, L_{\xi_b}\} f^i)$ and s_k is also the dimension of the fibers of S. Hence the surjectivity of ν requires that $D_{\xi_1,\cdots,\xi_k,f}$ be a full-rank matrix, namely that the s_k vectors $l_{a,b}$ are linearly independent among themselves. Since N is, by definition, orthogonal to $Tf(\mathcal{H})$ then the $l_{a,b}$ are also linearly independent from the $l_a = (L_{\xi_a} f^i)$; finally, the l_a are also all independent among themselves because f is, by hypothesis, a $\mathcal{H}$-immersion. Then the $k + s_k$ vectors $(l_a, l_{a,b})$ are linearly independent, i.e. f is $\mathcal{H}$-free. □

Theorem 3.1.11. *Let $\mathcal{H} \subset TM$ be a k-distribution on M, $\dim M = m$. The operator $\mathcal{D}_\mathcal{H}$ is infinitesimally invertible on the set of $\mathcal{H}$-free maps $f : M \to \mathbb{R}^q$. Moreover, if $q \geq m + k + s_k$, a generic map $M \to \mathbb{R}^q$ is $\mathcal{H}$-free.*

Proof. The infinitesimally invertibility of $\mathcal{D}_\mathcal{H}$ follows directly from the definition of $\mathcal{H}$-freedom. Observe that a map $f : M \to \mathbb{R}^q$ is $\mathcal{H}$-free when the image of the map

$$D_{\xi_1,\cdots,\xi_k,f} : M \to M_{s_k,q}(\mathbb{R})$$

is contained in the set of matrices of maximal rank. In particular a map is *not* $\mathcal{H}$-free when the image of $D_{\xi_1,\cdots,\xi_k,f}$ intersects the set $\mathcal{N}_{s_k,q}$ of matrices of non-maximal rank, whose codimension is $q + 1 - s_k$ [24]. For a generic f the image $D_{\xi_1,\cdots,\xi_k,f}(M)$ and $\mathcal{N}_{s_k,q}$ are transversal and therefore they do not have points in common when $\dim D_{\xi_1,\cdots,\xi_k,f}(M) < \operatorname{codim} \mathcal{N}_{s_k,q}$. Hence a generic map f is $\mathcal{H}$-free for $q > m - 1 + s_k$. □

Remark 3.1.12. *All concepts, statements and proofs of this section can be described naturally in the language of jet bundles, which we decided to avoid in order to keep our formalism as light as possible. We briefly point out below to the reader the definitions of $\mathcal{H}$-immersions and $\mathcal{H}$-free maps in this language.*

In analogy with the well known isomorphisms $J^1(M,N) \simeq T^*M \otimes TN$ and $J^2(M,N) \simeq (T^*M \oplus S^0_2 M) \otimes TN$ one can define the bundles

$$J^1(M,N;\mathcal{H}) := \mathcal{H}^* \otimes TN$$

and

$$J^2(M,N;\mathcal{H}) := (\mathcal{H}^* \oplus S^0_2\mathcal{H}^*) \otimes TN\,.$$

Then the $\mathcal{H}$-1-jet $j^1_{\mathcal{H}}f$ of a map $f : M \to \mathbb{R}^q$ is the section of the jet bundle

$$J^1(M,\mathbb{R}^q;\mathcal{H}) \to J^0(M,\mathbb{R}^q;\mathcal{H}) := M \times \mathbb{R}^q \to M$$

(whose fiber at the point (m,y) is $\mathcal{H}^*_m \otimes T_y\mathbb{R}^q$, i.e. matrices $k \times q$) given by

$$j^1_{\mathcal{H}}f(x^\alpha) = (x^\alpha, f^i(x), L_{\xi_a}f^i)\;.$$

A map f then is an $\mathcal{H}$-immersion iff $j^1_{\mathcal{H}}f(M)$ is contained in the set of maximal rank tensors. Similarly, the $\mathcal{H}$-2-jet $j^2_{\mathcal{H}}f$ is the section of

$$J^2(M,\mathbb{R}^q;\mathcal{H}) \to J^0(M,\mathbb{R}^q;\mathcal{H}) \to M$$

(whose fiber at the point (m,y) is $(\mathcal{H}^*_m \oplus S^0_2\mathcal{H}^*_m) \otimes T_y\mathbb{R}^q$, i.e. $(k+s_k)\times q$ matrices) given by

$$j^2_{\mathcal{H}}f(x^\alpha) = (x^\alpha, f^i(x), L_{\xi_a}f^i, \{L_{\xi_a}, L_{\xi_b}\}f^i)\;.$$

A map f then is $\mathcal{H}$-free iff $j^2_{\mathcal{H}}f(M)$ is contained in the set of maximal rank tensors of the bundle.

By combining Theorem 3.1.11 and the Newton-Nash-Moser-Gromov IFT (Theorem 2.2.8) we get as a corollary:

Theorem 3.1.13. *The restriction of $\mathcal{D}_{\mathcal{H}} : C^r(M,\mathbb{R}^q) \to \Gamma^0(S^0_2\mathcal{H})$ to $F^r_{\mathcal{H}}(M,\mathbb{R}^q)$ is an open map for every $r \geq 3$.*

3.1.1 Examples of $\mathcal{H}$-free maps in the critical dimension for distribution of dimension 1 and codimension 1

We show below a few concrete examples of $\mathcal{H}$-free maps $f : M \to \mathbb{R}^q$ for $q = k + s_k$ and either $\dim\mathcal{H} = 1$ or $\operatorname{codim}\mathcal{H} = 1$.

One-dimensional distributions

Let $\mathcal{H}$ be a 1-dimensional distribution, so that locally (and globally if $\mathcal{H}$ is orientable) $\mathcal{H}^1 = \mathrm{span}\{\xi\}$. Then, if ω is any 1-form such that $i_\xi\omega = 1$,

$$f^* e_q|_{\mathcal{H}} = \delta_{ij}\, L_\xi f^i\, L_\xi f^j\, \omega^2$$

and

$$D_{\xi,f} = \begin{pmatrix} L_\xi f^1 & \cdots & L_\xi f^q \\ L_\xi^2 f^1 & \cdots & L_\xi^2 f^q \end{pmatrix}$$

The condition for the existence of free maps reduces in this case to $q \geq 2$.

Example 3.1.14. *Take a regular vector field $\xi \in \mathfrak{X}(\mathbb{R}^m)$ and $\mathcal{H} = \mathrm{span}\{\xi\}$. Assume that ξ has a component always different from zero, e.g. $\xi^1 = 1$. Then $L_\xi x^1 = 1$ and a direct calculation shows that the function $F(x) = \psi(x^1)$ is $\mathcal{H}$-free for every free map $\psi : \mathbb{R} \to \mathbb{R}^2$. For example, one can take $\psi(t) = (t, e^t)$ or $\psi(t) = (\sin t, \cos t)$.*

Example 3.1.15. Consider the "constant slope" distribution on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

$$\mathcal{H} = \mathrm{span}\{\xi^x\partial_x + \xi^y\partial_y\} \subset T\mathbb{T}^2$$

If ξ^x and ξ^y are rationally dependent then we can transform ξ in, say, the constant vertical vector field with a global diffeomorphism and everything is trivial. Hence we assume that $\dim_{\mathbb{Q}}\{\xi^x, \xi^y\} = 2$, put $\theta = \xi^x/\xi^y$ and consider $\mathcal{H}$ as the span of $\xi' = \partial_x + \theta\partial_y$. Then every function $F(x, y) = \psi(x)$ and $G(x, y) = \psi(y)$ is $\mathcal{H}$-free for every $\psi \in F(\mathbb{S}^1, \mathbb{R}^2)$ since $L_{\xi'}x = 1 \neq 0$ and $L_{\xi'}y = \theta \neq 0$ (i.e. we use the same technique used in Theorem 3.2.3). For example the function $F(x, y) = (\sin x, \cos x)$ is $\mathcal{H}$-free.

Example 3.1.16. *Consider a regular vector field ξ on a Riemannian manifold (M, g) and assume that the 1-form $\xi^\flat$, "musical dual" of ξ, is exact. Then ξ is the gradient of some regular function f and $L_\xi f = \|\xi\|^2 > 0$ so that, as in the previous examples, $F(x) = \psi(f(x))$ is $\mathcal{H}$-free (for $\mathcal{H} = \mathrm{span}\{\xi\}$) for every free map $\psi : \mathbb{R} \to \mathbb{R}^2$.*

Example 3.1.17. *Consider the regular vector field $\xi = (y^2 - z^2 - 1, 2y, 2z) \in \mathfrak{X}(\mathbb{R}^3)$ inducing the distribution $\mathcal{H} = \mathrm{span}\{\xi\}$. This is the normal field to the level sets of $f(x, y, z) = e^x(y^2 + z^2 - 1)$ and therefore $L_\xi f > 0$ so that, again, every $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F = \psi(f)$ is $\mathcal{H}$-free for every $\psi \in F(\mathbb{R}^3, \mathbb{R}^2)$.*

Let us provide now an example of $\mathcal{H}$-free map to $\mathbb{R}^3$, i.e. when the matrix $\mathcal{D}_{\xi,f}$ is rectangular (notice that $\mathcal{H}^1$-free maps from $\mathbb{R}^3$ are generic starting with $q = 5$). Be α an unknown function from $\mathbb{R}^3$ to $\mathbb{R}$, define $F(x, y, z) = (x, e^x, \alpha(x, y, z))$ and set $g = y^2 + z^2$ to shorten formulas: then

$$\mathrm{rank}\, D_{\xi,F} = \mathrm{rank} \begin{pmatrix} g^2 - 1 & e^x(g^2 - 1) & L_\xi\alpha \\ g^2 - 1 + 4g^2 & e^x[(g^2 - 1)(g^2 - 1 + 4g^2)] & L_\xi^2\alpha \end{pmatrix}$$

The leftmost minor is invertible for $g \neq 1$ while for $g = 1$ the matrix reduces to

$$\begin{pmatrix} 0 & 0 & L_\xi \alpha \\ 4 & 4e^x & L_\xi^2 \alpha \end{pmatrix}$$

so it is enough to determine α so that $L_\xi \alpha|_{g=1} > 0$. Such a function is easily found by trial and error; an example is given by $\alpha(x, y, z) = y^2 + z^2$ since $L_\xi \alpha = 2g$. Hence the map $F(x, y, z) = (x, e^x, y^2 + z^2)$ is $\mathcal{H}$-free.

Codimension-1 distributions

Consider now a codimension-1 distribution $\mathcal{H}^{m-1} \subset TM^m$, so that locally it can be seen as the kernel of a regular 1-form ω or, equivalently, as the span of $m-1$ linearly independent vector fields $\{\xi_a\}$, $a = 1, \cdots, m-1$. In this case the metric induced by a map $f : M \to \mathbb{R}^q$ on $\mathcal{H}$ is given by

$$f^* e_q|_{\mathcal{H}} = \delta_{ij} L_{\xi_a} f^i L_{\xi_b} f^j \; \theta^a \otimes \theta^b$$

where $\{\omega, \theta^a\}$ is a base for $\Omega^1(M)$ such that $i_{\xi_a}\omega = 0$, $i_{\xi_a}\theta^b = \delta^b_a$.
The condition for the existence of free maps in this case reduces to $q \geq \frac{(m-1)(m+2)}{2}$; in particular, for $m = 3$, $\mathcal{H}$-free maps are generic for $q \geq 5$.

Example 3.1.18. *Take the two commuting vector fields $\xi_1 = (\cos y, -\sin y, 0)$ and $\xi_2 = (0, 0, 1)$ and consider the (integrable) distribution $\mathcal{H} = \mathrm{span}\{\xi_1, \xi_2\} \subset T\mathbb{R}^3$. The leaves of this $\mathcal{H}$ are the direct product of the level sets $f(x, y) = e^x \sin(y)$ with the z axis and, the space of leaves being not Hausdorff, this foliation is not topologically equivalent to the trivial one of $\mathbb{R}^3$. A direct computation gives that $L_{\xi_1}(e^x \cos y) = e^{2x} > 0$ and $L_{\xi_2} z = 1 > 0$ so that the map $f : \mathbb{R}^3 \to \mathbb{R}^5$ defined by*

$$f(x, y, z) = (\psi(e^x \cos y), \varphi(z), z e^x \cos y))$$

is $\mathcal{H}$-free for every pair of free maps $\psi, \varphi : \mathbb{R} \to \mathbb{R}^2$.

Example 3.1.19. *Consider the (non-integrable) distribution $\mathcal{H}^2 \subset T\mathbb{R}^3$ represented by the kernel of the canonical contact structure $\omega = y\,dx - dz$ or, equivalently, generated by the (non-commuting) vector fields $\xi_1 = (0, 1, 0)$ and $\xi_2 = (1, 0, -y)$ and take the function $f : \mathbb{R}^3 \to \mathbb{R}^5$ defined by*

$$f(x, y, z) = (y, x, e^y, e^x, z)$$

Here the matrix (3.4) writes:

$$D_{\xi_1, \xi_2, f} = \begin{pmatrix} 1 & 0 & e^y & 0 & 0 \\ 0 & 1 & 0 & e^x & -y \\ 0 & 0 & e^y & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & e^x & 0 \end{pmatrix}$$

whose determinant is $e^{x+y} > 0$, so that f is $\mathcal{H}$-free. Notice that here f is an immersion but it is not a free map since f_{zz} is identically zero.

3.2 Construction of $\mathcal{H}$-free maps for $q = k + s_k$

In this section we build free maps in critical dimension for three types of distributions. All three cases are inspired to the following result obtained by J.L. Weiner [28]:

Lemma 3.2.1 (Weiner). *Let h be a smooth function on $\mathbb{R}^2$ without critical points and let $\mathcal{H} = \ker dh \subset T\mathbb{R}^2$. Then there exists a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ whose level sets are transverse to $\mathcal{H}$ at every point.*

3.2.1 One-dimensional distributions on $\mathbb{R}^2$

In our first generalization of Lemma W we weaken the hypothesis $\mathcal{H} = \ker dh$, i.e. we do not assume anymore that $\mathcal{H}$ is a Hamiltonian foliation and get the following:

Lemma 3.2.2. *Let $\mathcal{H}$ be any 1-dimensional distribution on $\mathbb{R}^2$ of finite type. The following three (equivalent) properties hold true:*

1. *there exists a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ whose level sets are transverse to $\mathcal{H}$ at every point;*
2. *for any regular 1-form ω such that $\mathcal{H} = \ker \omega$ there exists a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $*(\omega \wedge df) > 0$;*
3. *for any regular section ξ of $\mathcal{H}$ there exists a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $L_\xi f > 0$.*

Since this result involves regular vector fields on the plane, which will be thoroughly studied in next chapter, we postpone to it the relative definitions and proof of the Lemma (see Definition 4.0.17 and Propositions 4.2.1 and 4.2.3).

With this we are in condition to easily prove the following:

Theorem 3.2.3. *Let $\mathcal{H} \subset T\mathbb{R}^2$ be a one-dimensional distribution of finite type on $\mathbb{R}^2$. Then there exists a smooth function $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ such that $\psi \circ f \in F^r_{\mathcal{H}}(\mathbb{R}^2, \mathbb{R}^2)$ for all $\psi \in F^r(\mathbb{R}, \mathbb{R}^2)$.*

Proof. Let ξ be a regular section of the distribution $\mathcal{H} \subset T\mathbb{R}^2$, $\mathcal{H} = \ker \omega$ such that $i_\xi(*\omega) = 1$ and let $U = L_\xi^{-1}\left(C^\infty_+(\mathbb{R}^2)\right)$ denote the set of all smooth real valued maps f on $\mathbb{R}^2$ such that $L_\xi f > 0$ (this set is non-empty by the previous lemma). We want to show that, for every free map $\psi : \mathbb{R} \to \mathbb{R}^2$, the map

$$F(x, y) = \psi(f(x, y))$$

is $\mathcal{H}$-free. Take $\psi(t) = (a(t), b(t))$. We must verify that the matrix

$$\begin{pmatrix} L_\xi[a(f)] & L_\xi[b(f)] \\ L^2_\xi[a(f)] & L^2_\xi[b(f)] \end{pmatrix}$$

has rank 2 (cfr. Example 3.1.9). A direct computation shows that

$$\det D_{\xi,F} = \begin{vmatrix} L_\xi[a(f)] & L_\xi[b(f)] \\ L_\xi^2[a(f)] & L_\xi^2[b(f)] \end{vmatrix} =$$

$$= \begin{vmatrix} a'(f)L_\xi f & b'(f)L_\xi f \\ a'(f)L_\xi^2 f + a''(f)[L_\xi f]^2 & b'(f)L_\xi^2 f + b''(f)[L_\xi f]^2 \end{vmatrix} =$$

$$= [a'(f)b''(f) - b'(f)a''(f)][L_\xi f]^3 \neq 0$$

which, by hypothesis, is never zero. In fact we assumed $L_\xi f > 0$ and the first factor cannot vanish since ψ is free. □

Example 3.2.4. *Consider the distribution $\mathcal{H} \subset \mathbb{R}^2$ defined by*

$$\mathcal{H} = \operatorname{span}\{\xi = 2y\partial_x + (1 - y^2)\partial_y\}\,.$$

This $\mathcal{H}$ is of finite type since it has only a pair of separatrices, the straight lines $y = \pm 1$ (see Definition 4.0.17). Indeed $\mathcal{H}$ is the tangent space to the (Hamiltonian) foliation $\mathcal{F}$ of the level sets of $f(x, y) = (y^2 - 1)e^x$ (a direct computation shows that $L_\xi((y^2 - 1)e^x) = 0$).

Moreover, $L_\xi(ye^x) = (1 + y^2)e^x > 0$, i.e. the foliation of the level sets of the function $g(x, y) = (1 + y^2)e^x$ is transverse to $\mathcal{F}$ at every point (cfr. Fig. 4.1). Then, by Theorem 3.2.3, $F(x, y) = \psi(g(x, y))$ is $\mathcal{H}$-free for every free map $\psi : \mathbb{R} \to \mathbb{R}^2$. For example, $F(x, y) = (ye^x, e^{ye^x})$ is $\mathcal{H}$-free (cfr. Example 3.1.14).

3.2.2 The case of completely integrable systems

In our second generalization of Lemma W we reinterpret it in terms of completely integrable systems.

Let (M^{2n}, Ω) be a connected symplectic manifold. Since the symplectic 2-form is non-degenerate it sets up a linear isomorphism between vector fields ξ and 1-forms ω on M through the relation $i_\xi \Omega = \omega$. Moreover, every real valued function $f : M \to \mathbb{R}$ determines a unique vector field ξ_f called *Hamiltonian vector field* with the *Hamiltonian* f by requiring that for every vector field η on M the identity $df(\eta) = \omega(\xi_f, \xi_\eta)$ must hold. To the given symplectic structure Ω we can associate, in a natural way, the *Poisson bracket* via the formula $\{f, g\} = \Omega(\xi_f, \xi_\eta)$ which turns the algebra $C^\infty(M)$ of smooth functions on M into a Poisson algebra. Assume that (M^{2n}, Ω) admits a regular completely integrable system. This means that there exists a maximal set of functionally independent Poisson commuting functions $\{I_i\}$, i.e., such that $dI_1 \wedge \cdots \wedge dI_n \neq 0$ at every point of M and that the Poisson subalgebra generated by the I_i in $C^\infty(M)$ is abelian. Consider the distribution $\mathcal{H} = \ker dI_1 \cap \cdots \cap \ker dI_n$ and the corresponding Lagrangian foliation $\mathcal{F}$ (so that $\mathcal{H} = T\mathcal{F}$). Then the following theorem, classically known as *Arnold–Liouville theorem* holds true (see [29] or [30] for details).

Theorem 3.2.5 (Arnold–Liouville)**.** *Let $\mathcal{F}$ the Lagrangian foliation defined above. If every Hamiltonian vector field ξ_{I_i} is complete then every leaf of $\mathcal{F}$ is diffeomorphic to $\mathbb{T}^r \times \mathbb{R}^{n-r}$ and has a saturated neighbourhood U (with respect to the projection onto the space of leaves $\mathcal{F}$) symplectomorphic to the product manifold $D \times (\mathbb{T}^r \times \mathbb{R}^{n-r})$, where $D \subset \mathbb{R}^n$ is open, endowed with the coordinates (I_i, φ^j) and with the canonical symplectic form $\Omega_0 = dI_i \wedge d\varphi^i$.*

This statement means, in particular, that the commutation relations between the special coordinates (I_i, φ^j) are given by the well-known

$$\{I_i, I_j\} = 0\,, \;\; \{\varphi^i, \varphi^j\} = 0\,, \;\; \{I_i, \varphi^j\} = \delta_i^j.$$

The (I_i, φ^j) are usually called "action-angle" coordinates.

When $M = \mathbb{R}^2$ every Hamiltonian system (represented by a single Hamiltonian) is, trivially, a completely integrable system. In particular, Lemma W can be restated as follows:

Lemma 3.2.6 (Weiner)**.** *Let $\{I\}$ be a regular completely integrable system on the symplectic manifold $(\mathbb{R}^2, \Omega_0 = dx \wedge dy)$. Then there exists a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\Omega_0(\xi_I, \xi_f) > 0$ for all points of $\mathbb{R}^2$.*

In the following Lemma we extend Weiner's one to higher dimensional integrable systems:

Lemma 3.2.7. *Let $\{I_1, \cdots, I_n\}$ be a regular completely integrable system on (M^{2n}, Ω) and suppose that all the Hamiltonian vector fields ξ_{I_i} are complete. Then there exist n smooth functions $\{f^1, \cdots, f^n\}$ (possibly multi-valued) such that*

$$\{I_i, f^i\} > 0\,, \;\; \{I_i, f^j\} = 0\,, \; j \neq i \tag{3.5}$$

on the whole manifold M^{2n}.

Proof. We follow closely the original argument in [28]. By Arnold-Liouville Theorem, every leaf $l \in \mathcal{F}$ has a saturated neighbourhood $U_l \simeq \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{T}^{n-k}$ with coordinates (I_i, φ_l^j) such that U_l is defined by the inequalities $\alpha_i^l \le I_i \le \beta_i^l$ and

$$\{I_i, \varphi_l^j\} = \delta_i^j.$$

We renormalize the action coordinates I_i (which, by hypothesis, are global) by $J_i^l = \mu_i^l (I_i - \nu_i^l)$ so that U_l is characterized as the connected component of $|J_i^l| < 2$ containing l. Now, let $b : \mathbb{R} \to \mathbb{R}$ be any bump function with support equal to $(-1, 1)$ and let $l : \mathbb{R} \to \mathbb{R}$ be any smooth non-decreasing function which is equal to 0 on $(-\infty, -1]$, to 1 on $[1, \infty)$ and strictly increasing between 0 and 1 on $(-1, 1)$.

The functions defined on U_l as

$$f_l^j = b(J_1^l) \cdots b(J_n^l) l(\varphi_l^j)$$

can be trivially extended to the whole M by setting $f_l^j = 0$ outside U_l. Note that their differentials

$$df_l^i = \sum_{i=1}^{\infty} \mu_i^l b'(J_i^l) dI_i + \sum_{i=1}^{\infty} b(J_n^l) l'(\varphi_l^i) d\varphi_l^i$$

have (modulo the span of the dI_i) compact support

$$V_l = \{p \in U_l | \, |J_i^l(p)| < 1 \, , \; |\varphi_l^j(p)| < 1\} \, .$$

Moreover, we have that

$$\{I_i, f_l^i\} = b(J_1^l) \cdots b(J_n^l) l'(\varphi_l^i) > 0 \, , \;\; \{I_i, f_l^j\} = 0 \, , \;\; j \neq i$$

inside V_l while all Poisson brackets are identically zero outside V_l.

We extract from the covering $\{U_l\}$ a countable subcovering $\{U_{l_k}\}$. and show that, by a convenient choice of the coefficients a_k, the series

$$f^i = \sum_{k \in \mathbb{N}} a_k f_{l_k}^i$$

can be made convergent. In fact the $f_{l_k}^i$ are uniformly bounded so that, by taking $a_k = 2^{-k}$, the series can be made uniformly convergent. Next, let us fix any n-dimensional distribution $\mathcal{H}'$ transverse to $\mathcal{F}$ and consider on M the Riemannian metric $g = \sum_{i=1}^n (dI_i)^2 + g'$ (where g' is any metric on $\mathcal{H}'$). Denote by $\|D^{(j)} f_{l_k}^i\|$ the norm (associated to the metric g) of the derivatives of order j of $f_{l_k}^i$. This is seen as a map with domain M and range the symmetric product bundle (of order j) $S^j M$ based on M. We thus get that, for every value of k, there is some finite constant M'_k such that, outside V_{l_k}, $\|D^{(j)} f_{l_k}^i\| \leq M'_{k,i}$ for $1 \leq j \leq k$. Since V_{l_k} has compact closure, there exists another constant $M''_{k,i}$ such that $\|D^{(j)} f_{l_k}^i\| \leq M'_{k,i}$ within V_{l_k}. This means that $\|M_{k,i}^{-1} D^{(j)} f_{l_k}^i\| \leq 1$ for $M_{k,i} = \min\{1, M'_{k,i}, M''_{k,i}\}$. Therefore, if we take $a_k = 2^{-k} M_{k,i}^{-1}$, the series $\sum_{k \in \mathbb{N}} a_k D^{(j)} f_{l_k}^i$ uniformly converges for each $j \in \mathbb{N}$.

Then the f^i are smooth and one has

$$\{I_i, f^i\} > 0 \, , \;\; \{I_i, f^j\} = 0 \, , \;\; j \neq i.$$

Finally, Arnold–Liouville's Theorem tells that the neighbourhoods U_l are all symplectomorphic to $\mathbb{R}^n \times (\mathbb{T}^r \times \mathbb{R}^{n-r})$ for some r between 0 and n and, for $r > 0$ the leaves have compact components. Observe that, on these components, the df^j are well-defined closed 1-forms. Nevertheless, these forms may be non-exact, due to the non-triviality of the first cohomology group of the leaves. Consequently, in this case the functions f^j may be multivalued, namely well-defined only on some covering of M. □

Theorem 3.2.8. *Let (M^{2n}, Ω) be a symplectic manifold admitting a completely integrable system $\{I_1, \cdots, I_n\}$, $\mathcal{H} \subset TM$ the n-dimensional Lagrangian distribution $\mathcal{H} = \cap_{i=1}^n \ker dI_i$ and $\mathcal{F}$ the corresponding Lagrangian foliation. Assume* *that the Hamiltonian vector fields associated to the I_i are all complete and that every leaf of $\mathcal{F}$ has no compact component. Then it is possible to find n smooth real valued functions f^i, $i = 1, \dots, n$, on M such that the map $F : M \to \mathbb{R}^{n+s_n}$, $s_n = \frac{n(n+1)}{2}$ defined by*

$$F(x) = (\psi_1(f^1(x)), \cdots, \psi_n(f^n(x)), f^1(x)f^2(x), \cdots, f^{n-1}(x)f^n(x))$$

is $\mathcal{H}$-free for any choice of n free maps $\psi_i : \mathbb{R} \to \mathbb{R}^2$.

Proof. By Lemma 3.2.7 there exist n smooth functions f^i satisfying (3.5) which, since the leaves of the foliation $\mathcal{F}$ have no compact component, are all single-valued. We consider n free maps $\{\psi_1, \cdots, \psi_n\}$ from $\mathbb{R}$ to $\mathbb{R}^2$ and prove that the map $F : M \to \mathbb{R}^{n+s_n}$ defined as

$$F(x) = (\psi_1(f^1(x)), \cdots, \psi_n(f^n(x)), f^1(x)f^2(x), \cdots, f^{n-1}(x)f^n(x))$$

is $\mathcal{H}$-free.

Let $\psi_i(t) = (a_i(t), b_i(t))$ and set $D\psi_i = a_i' b_i'' - a_i'' b_i'$. The square matrix $D_{\xi_1, \cdots, \xi_n, F}$ (see (3.4) above) is given, up to a permutation of its rows, by

$$\left(\begin{array}{c|c|c|c|c|c}
A_1 & * & * & * & * & * \\ \hline
0 & \ddots & * & * & * & * \\ \hline
0 & 0 & A_n & * & * & * \\ \hline
0 & 0 & 0 & 2g_1g_2 & * & * \\ \hline
0 & 0 & 0 & 0 & \ddots & * \\ \hline
0 & 0 & 0 & 0 & 0 & 2g_{n-1}g_n
\end{array}\right)$$

where $g_i = L_{\xi_{I_i}} f^i$,

$$A_i = \begin{pmatrix} a_i'(f^i) g_i & b_i'(f^i) g_i \\ a_i'(f^i) L^2_{\xi_{I_i}} f^i + a_i''(f^i) g_i^2 & b_i'(f^i) L^2_{\xi_{I_i}} f^i + b_i''(f^i) g_i^2 \end{pmatrix}$$

and the stars represent terms which do not contribute to the determinant.

Since $\det A_i = g_i^3 D\psi_i$ and the blocks below the diagonal are identically zero, the determinant of $D_{\xi_1, \cdots, \xi_n, F}$ equals

$$2^{s_n} \Pi_{k=1}^n (g_i^{n+2} D\psi_k)$$

which differs from zero at every point because, by construction, $g_i > 0$ and, by hypothesis, $D\psi_i \neq 0$. Hence F is a $\mathcal{H}$-free map. □

Remark 3.2.9. *Clearly the map F defined in the proof above is modeled after the canonical free map $G : \mathbb{R}^n \to \mathbb{R}^{n+s_n}$ given by*

$$G(x^1, \dots, x^n) = (x^1, \dots, x^n, (x^1)^2, x^1x^2, \cdots, (x^n)^2) \, .$$

So far it is not known (see [1], p.9, and Section **??** *of this thesis) whether, for $n \geq 3$, there exist free maps from $\mathbb{T}^n$ to $\mathbb{R}^{n+s_n}$. It is for this reason that* *in Theorem 3.2.8 we require that the leaves of the foliation $\mathcal{F}$ have no compact component. On the other hand, it is an easy matter to check that the map $G : \mathbb{T}^n \to \mathbb{R}^{n+s_n+s_{n-1}}$ defined by*

$$G(\theta^1, \dots, \theta^n) = (\mathrm{cs}(\theta^1), \dots, \mathrm{cs}(\theta^n), \mathrm{cs}(\theta^1 + \theta^2), \dots, \mathrm{cs}(\theta^{n-1} + \theta^n)) ,$$

where $\mathrm{cs}\,\theta = (\cos\theta, \sin\theta)$, is free. When the leaves of $\mathcal{F}$ are compact (and therefore diffeomorphic to $\mathbb{T}^n$), the map $F : M \to \mathbb{R}^{n+s_n+s_{n-1}}$ defined by $F(x) = G(f^1(x), \dots, f^n(x))$ is $\mathcal{H}$-free. Unlike the case of Theorem 3.2.8, the dimension of the target space of F is not the smallest possible for a $\mathcal{H}$-free map (except for $n = 1$). Nevertheless the existence of such an F is a non-trivial fact since Theorem 3.1.11 grants the existence of $\mathcal{H}$-free maps from M to $\mathbb{R}^q$ only for $q \geq 3n + s_n$ and, for $n \leq 4$, we have that $n + s_n + s_{n-1} < 3n + s_n$.

3.2.3 The case of Poisson systems

In our last generalization we reinterpret Weiner's Lemma in terms of Poisson geometry.

Poisson structures are a generalization of symplectic structures having the nice property of existing even in odd-dimensional manifolds. Recall that a Poisson manifold is a pair $(M, \{,\})$ where $\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is a $\mathbb{R}$-bilinear skew-symmetric derivation satisfying the Jacobi identity. To every smooth function $f \in C^\infty(M)$ it is associated canonically a *Hamiltonian vector field* ξ_f defined by $\xi_f(g) \stackrel{\text{def}}{=} \{f, g\}$.

In particular, when $M = \mathbb{R}^2$, the canonical symplectic form $\Omega_0 = dx \wedge dy$ induces on M a Poisson Bracket $\{f, g\} = \Omega_0(\xi_f, \xi_g)$ which can also be obtained via the Euclidean metric as

$$\{f, g\} = *[df \wedge dg]$$

where $*$ is the Euclidean Hodge operator. Observe that this Poisson bracket does not need a symplectic structure to be defined but rather an orientable Riemannian structure. Furthermore, it can be defined in any dimension n as follows. Let M be an oriented Riemannian manifold of dimension $n \geq 2$, $*$ its Hodge operator and $H = \{h_1, \cdots, h_{n-2}\}$ a set of $n-2$ smooth functions. We set

$$\{f, g\}_H \stackrel{\text{def}}{=} *[dh_1 \wedge \cdots \wedge dh_{n-2} \wedge df \wedge dg]$$

and call it *Riemann-Poisson* bracket with respect to H. In particular, the foliation corresponding to a Hamiltonian vector field ξ_h, with $h \in C^\infty(M)$, is given by the intersections of the level sets of the h_i with the level sets of h.

Remark 3.2.10. *Each function h_i in H is a Casimir for $\{,\}_H$. In particular, the foliation corresponding to a Hamiltonian vector field ξ_h is given by the intersections of the level sets of the h_i with the level sets of h.*

Example 3.2.11. *Let $M = \mathbb{R}^3$ with the Euclidean metric and coordinates (x, y, z) and let $H = \{x\}$. Then the Riemann-Poisson bracket is given by*

$\{f, g\}_H = \partial_y f \, \partial_z g - \partial_y g \, \partial_z f$. In particular $\xi_y = \partial_z$ and $\xi_z = -\partial_y$ and the coordinate x is a Casimir.

Example 3.2.12. *Let $M = \mathbb{T}^3$ with angular coordinates $(\theta^1, \theta^2, \theta^3)$ and $H = \{h(\theta^i) = B_i \theta^i\}$, $i = 1, 2, 3$, for some constant 1-form $B = B_i d\theta^i$. Note that h is a multi-valued function; this is allowed because in the definition of the bracket appear only the derivatives of h rather than h itself. Then the Riemann-Poisson bracket is given by*

$$\{f, g\}_H = \epsilon^{ijk} \partial_i f \, \partial_i g \, B_k \ ,$$

where ϵ^{ijk} is the totally antisymmetric Levi–Civita tensor.

This bracket was introduced by S.P. Novikov as an application of his generalization of Morse theory to multivalued functions [31]. An example of the rich topological structure hidden in this Riemann-Poisson bracket can be found in [32].

Placed in this setting Weiner's Lemma reads as follows:

Lemma 3.2.13. *Consider the Euclidean plane $\mathbb{R}^2$ endowed with the Riemann-Poisson bracket $\{,\}$ and let $h \in C^\infty(\mathbb{R}^2)$ be a regular Hamiltonian. Then there exists $f \in C^\infty(\mathbb{R}^2)$ such that $\{h, f\} > 0$ on the whole $\mathbb{R}^2$.*

The next lemma allows, under a non-degeneracy condition, to extend Weiner's result in the latter formulation to Riemann-Poisson brackets in higher dimension. The proof of this Lemma goes along the same lines of the original Weiner's proof.

Lemma 3.2.14. *Let M be an oriented Riemannian manifold of dimension $n \geq 2$ and let $H = \{h_1, \cdots, h_{n-2}\}$ be a set of $n-2$ functions functionally independent at every point (i.e. such that $dh_1 \wedge \cdots \wedge dh_{n-2}$ never vanishes). Then, for any $h \in C^\infty(M)$ functionally independent from the h_i, there exists a smooth function (possibly multivalued) $f : M \to \mathbb{R}$ such that the Riemann–Poisson bracket $\{h, f\}_H$ is strictly positive at every point.*

Proof. Set $h_{n-1} = h$. Let $\mathcal{F}$ the 1-dimensional Hamiltonian foliation associated to h, namely the one defined by $dh_1 = \cdots = dh_{n-1} = 0$, and let $\pi : M \to \mathcal{F}$ be the canonical associated projection. At every point $p \in M$ there exists a saturated (with respect to π) neighbourhood $U_p \simeq D \times X$, where $D \simeq \mathbb{R}^{n-1}$ and X is either $\mathbb{R}$ or $\mathbb{S}^1$, defined as the connected component of the set $W_p = \{a_i < h_i < b_i, i = 1, \ldots, n-1\}$ which contains p. We renormalize these coordinates by using new ones $\hat{h}_i = \mu_i(h_i - \nu_i)$ so that W_p is defined by $|\hat{h}_i| < 2$, $i = 1, \ldots, n-1$.

Let now A_p be the subset of U_p defined by $|\hat{h}_i| < 1$, $i = 1, \ldots, n-1$ and take two functions b and l like in the proof of Lemma 3.2.7.

The real-valued function

$$f_p(h_1, \ldots, h_{n-1}, \varphi) = l(\varphi) \times \prod_{i=1,\ldots,n-1} b(h_i)$$

is well-defined and smooth in A_p and it can be extended to a smooth function on the whole M by setting it equal to zero outside A_p. Clearly

$$df_p = \omega_{n-1} \oplus \Pi_{i=1}^{n-1} b(h^i) l'(\varphi) d\varphi$$

where $\omega_{n-1} \in \text{span}\{dh_1, \cdots, dh_{n-1}\}$. Then $\{h, f_p\}_H$ everywhere vanishes except within $B_p = \{p' \in A_p | \, |\varphi(p')| < 1\}$, where we have

$$\begin{aligned}\{h, f_p\}_H &= *[dh_1 \wedge \cdots \wedge dh_{n-1} \wedge df_p] \\ &= *[dh_1 \wedge \cdots \wedge dh_{n-1} \wedge \Pi_{i=1}^{n-1} b(h_i) l'(\varphi) d\varphi] \\ &= *[dh_1 \wedge \cdots \wedge dh_{n-1} \wedge d\varphi] \prod_{i=1}^{n-1} b(h_i) l'(\varphi) > 0\,,\end{aligned}$$

the function $*[dh_1 \wedge \cdots \wedge dh_{n-1} \wedge d\varphi]$ being positive for all points $q \in U_p$ and every $p \in M$ since M is oriented.

Now extract a countable subcovering $\{A_{p_k}\}_{k\in\mathbb{N}}$ from $\{A_p\}$ and let $f_k = f_{p_k}$ be the corresponding function on every $A_k := A_{p_k}$. As in Lemma 3.2.7, the series $\sum_k a_k f_k$ can be made convergent to a smooth function f by choosing a convenient sequence a_k. Then $\{h, f\}_H > 0$ on the whole M since every point p is covered by at least one A_k, so that $\{h, f\}_H \geq \{h, f_k\}_H > 0$. □

Theorem 3.2.15. *Let M be an n-dimensional oriented Riemannian manifold, $H = \{h_1, \cdots, h_{n-2}\}$ a set of $n-2$ functions functionally independent at every point and $\{\cdot,\cdot\}_H$ the corresponding Riemann-Poisson bracket. If $h \in C^\infty(M)$ is functionally independent from all the h_i and $\mathcal{H}$ is the corresponding Hamiltonian 1-dimensional distribution, then there exists a (possibly multivalued) smooth function $f : M \to \mathbb{R}$ such the smooth map $F : M \to \mathbb{R}^2$ given by $F(x) = \psi(f(x))$ is $\mathcal{H}$-free for every free map $\psi : A \to \mathbb{R}^2$ where $A = \mathbb{R}$ if f is single-valued or $A = \mathbb{S}^1$ if f is multivalued.*

Proof. Let ξ_h be the Hamiltonian vector field associated to h through $\{,\}_H$. Then, by Lemma 3.2.14, there exists a function f (possibly multi-valued) such that $L_\xi f > 0$. Hence, as seen in Theorem 3.2.3, the smooth map $F : M \to \mathbb{R}^2$ given by $F(x) = \psi(f(x))$ is $\mathcal{H}$-free ($\mathcal{H} = \text{span}\{\xi_h\}$), where $\psi : \mathbb{R} \to \mathbb{R}^2$ (respectively $\psi : \mathbb{S}^1 \to \mathbb{R}^2$) is free if f is single-valued (respectively multi-valued). □

4

Cohomological equation in the plane

The study of planar vector fields has a long history. The first to address the problem of the qualitative study of global solutions of ODEs in $\mathbb{R}^2$ was Poincaré, in a series of papers published between 1880 and 1882 (see [33]). These papers represent the beginning of the whole renown qualitative theory of dynamical systems, which was initiated by Poincaré as part of his program of solving the three body-problem. (see [34] and [35] for more details and bibliography on this topic). In 1900 Hilbert [36] pointed out the high non-triviality of the problem of the classification of plane vector fields by proposing as his sixteenth problem, still unsolved, the evaluation of the number of limit cycles of a polynomial plane vector fields. In Thirties Whitney realized that the subset of *regular* planar vector fields, i.e. those without zeros, is much more treatable and started studying them [37, 38, 39]. A complete classification of regular vector fields on the plane was done by his pupil Kaplan [27, 40] using an ad-hoc topological tool (chordal systems). In this thesis we rather use the more general concept of *inseparable leaves* and *separatrices*, introduced by L. Markus [41] while working at the extension of Kaplan's results to the more general problem of the topological classification of all planar vector fields.

We recall a few standard basic concepts and definitions that will be used in this chapter. We denote by $\mathfrak{X}_r(\mathbb{R}^2)$ the set of all smooth regular vector fields in the plane, by $\mathcal{F}_\xi$ the foliation of the integral trajectories of $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$ and by $\pi_\xi : \mathbb{R}^2 \to \mathcal{F}_\xi$ the canonical projection that sends every point in the leaf[1] passing through it. We endow $\mathcal{F}_\xi$ with the canonical quotient topology. It was shown by Haefliger and Reeb [42] that $\mathcal{F}_\xi$ admits the structure of a 1-dimensional simply connected second countable non (necessarily) Hausdorff smooth manifold; the smooth structure is characterized by the property that the restriction of π_ξ to every transversal line ℓ is a diffeomorphism onto its image. Two integral trajectories s_i, $i = 1, 2$, of ξ are said *inseparable* when their projections $\pi_\xi(s_i)$

[1] Throughout the thesis we refer to the points of $\mathcal{F}_\xi$ as *integral trajectories* or *leaves* depending on the aspect of them we want to emphasize.

cannot be separated in the topology of $\mathcal{F}_\xi$ (e.g. see Fig. 4.1). We denote by $\mathcal{I}_{\mathcal{F}_\xi,s}$ the set of all leaves distinct from s inseparable from it (note that $\mathcal{I}_{\mathcal{F}_\xi,s}$ is empty for all but countably many leaves) and by $\mathcal{S}_{\mathcal{F}_\xi}$ the (countable) set of leaves for which $\mathcal{I}_{\mathcal{F}_\xi,s}$ is not empty. A leaf s is called a *separatrix* when the boundary of every neighbourhood of $\pi_\xi(s)$ contains more than two points. The set of all separatrices is the closure of $\mathcal{S}_{\mathcal{F}_\xi}$ [41]. In the present thesis we will rather use the term separatrix to indicate just the elements of $\mathcal{S}_{\mathcal{F}_\xi}$ since their limit points play no role in our work. Every plane foliation is orientable and, correspondingly, to each set $\mathcal{I}_{\mathcal{F}_\xi,s}$ can be given a natural order; we say that two separatrices are *adjacent* if they are next to each other with respect to this order.

We introduce now a few specific definition we will need throughout the chapter.

Definition 4.0.16. *Two vector fields ξ and ξ' are* strongly proportional *if they are proportional through a non-zero smooth function. A vector field ξ is* intrinsically Hamiltonian *if it is strongly proportional to a Hamiltonian vector field and is* transversally Hamiltonian *if it is transversal to a* Hamiltonian foliation *$\mathcal{G}$, i.e. to the level sets of a regular smooth function G (we say that G is a Hamiltonian for G).*

It is easily seen that a regular vector field is intrinsically Hamiltonian iff the PDE $L_\xi f = 0$ admits a regular smooth solution and is transversally Hamiltonian iff is is solvable the differential inequality $L_\xi f > 0$.

Definition 4.0.17. *A foliation $\mathcal{F}_\xi$ (or simply the vector field ξ) is of* finite type *if $\mathcal{S}_{\mathcal{F}_\xi}$ is closed and every set $\mathcal{I}_{\mathcal{F}_\xi,s}$ is finite.*

In this case the complement of the set of separatrices is the disjoint union of countably many unbounded open sets named by Markus [41] *canonical regions* and the boundary of each canonical region has a finite number of connected components. We recall that examples of smooth or even analytic foliations of the plane with a dense set of separatrices are known in literature (see [43] and [44]). While there are reasons to believe that such foliations are generic in some "combinatorial" sense, the set of foliations of finite type is nevertheless of great importance since important natural categories of regular vector fields leads to them. For example every polynomial vector field is of finite type: finite bounds for the number of the inseparable leaves of a polynomial vector field were find first by Markus [45] and later improved independently by M.P. Muller [46] and S. Schecter and M.F. Singer [47]. It is easy to verify that are of finite type also all regular vector fields strongly proportional to those of the kind $\xi(x, y) = (a(y), b(y))$, where (a, b) is a generic pair of Morse functions of one variable (so that $a^2 + b^2$ is strictly positive).

Definition 4.0.18. *A* complete set of transversals *(CST) for $\mathcal{F}_\xi$ is a set of lines $\mathcal{T}_\xi = \{\ell_i\}$, one for each separatrix of $\mathcal{F}_\xi$, such that every ℓ_i is transversal to $\mathcal{F}_\xi$ and cuts the corresponding separatrix s_i and the set $\{\pi_\xi(\ell_i)\}$ covers $\mathcal{F}_\xi$.*

We call gap *of* $g \in C^\infty(\mathbb{R}^2)$ *between two adjacent separatrices* s_1 *and* s_2 *with respect to the CST* $\mathcal{T}_\xi$ *the limit (if it exists)*

$$\underset{\mathcal{T}_\xi}{gap}(g; s_1, s_2) = \lim_{p \to p_1} \int_0^{T_p} g\left(\Phi^t_\xi(p)\right) dt\,,$$

where the point $p \in \ell_1$ *tends to* $p_1 = \ell_1 \cap s_1$*,* Φ^t_ξ *is the flux of* ξ *and* T_p *is the unique number s.t.* $\Phi^{T_p}_\xi(p) \in \ell_2$[2]

Finally we set a few notations on spaces of germs we are going to use in the last section. Let $a \in \mathbb{R}$. We denote by H^r_a the ring of left germs at a of functions in $C^r(-\infty, a)$, i.e. the equivalence classes determined by the equivalence relation $h \simeq h'$ if h and h' coincide in some interval of the form $(a-\epsilon, a)$ for some $\epsilon > 0$, and by G^r_a the subring of the left germs in H^r_a which can be extended to a continuous function at a together with their derivatives up to order r. Similarly, let $I = \{a\} \times [b_1, b_2]$ and set $L_I = (-\infty, a] \times \mathbb{R} \setminus I$. We denote by H^r_I the ring of left germs at I of functions of $C^r(L_I)$, i.e. $h \simeq h'$ if h and h' coincide in some set $(U \cap L_I) \setminus I$, where U is a neighbourhood of I, and by G^r_I the subring of germs of functions of H^r_I which can be extended to C^r functions on the whole L_I.

Definition 4.0.19. *We call* singular left germs at $a \in \mathbb{R}$ *the elements of the quotient ring* $SG^r_a = H^r_a/G^r_a$ *and* singular left germs at $I = \{a\} \times [b_1, b_2]$ *the elements of the quotient ring* $SG^r_I = H^r_I/G^r_I$*.*

Note that in this chapter we are interested only to the action of ξ on smooth functions; concerning the global solvability in other functional spaces, e.g. of entire functions or Gevrey-type functions in the realm of global Cauchy-Kowalevskaya theorem see [48, 49] and the references therein. Besides look next chapter for a study of the type of singularities that can arise at separatrices when we weaken the regularity conditions on the rhs.

The results presented in this chapter will be published in [4].

4.1 coker L_ξ

As pointed out above in Theorem 1.3.1, if $\mathcal{F}_\xi$ admits a global transversal the method of characteristics provides a global solution to the cohomological equation (1.1) for every $g \in C^\infty(\mathbb{R}^2)$, so that $L_\xi(C^\infty(\mathbb{R}^2)) = C^\infty(\mathbb{R}^2)$ and $\operatorname{coker} L_\xi = \{0\}$. The obstruction to the existence of global transversals is the presence of separatrices since no smooth line ℓ can, at the same time, be transversal to $\mathcal{F}_\xi$ and intersect any pair leaves inseparable from each other.

In absence of global transversals, one can try to solve $L_\xi f = g$ recursively in the following way. Let s be a separatrix for ξ and denote by ℓ any transversal through it and by $U_\ell = \pi^{-1}_\xi(\ell) \subset \mathbb{R}^2$ the saturated open set containing ℓ.

[2] Such number exists for s_1 and s_2 are inseparable and is unique for every transversal cuts each leaf at most once.

Since U_ℓ is a proper subset of $\mathbb{R}^2$, its boundary is non-empty and equal to the union of the sets $\mathcal{I}_{\mathcal{F}_\xi,\tilde{s}}$ corresponding to all leaves $\tilde{s}$ cut by ℓ. By construction ξ, once restricted to U_ℓ, admits a global transversal (the line ℓ) and therefore $L_\xi(C^\infty(U_\ell)) = C^\infty(U_\ell)$. Let now g_ℓ be any solution, in U_ℓ, to $L_\xi f = g$. We can try to extend g_ℓ beyond U_ℓ by selecting any boundary component s' of ∂U_ℓ and any transversal ℓ' passing through it. The function g_ℓ restricts on $\ell' \cap U_\ell$ to a smooth function $\hat{g}_{\ell'}$; if we can extend $\hat{g}_{\ell'}$ to a smooth function $g_{\ell'}$ defined on the whole ℓ' then, via the method of characteristics applied to the set $U_{\ell'} = \pi_\xi^{-1}(\ell')$ and using $g_{\ell'}$ as initial condition on ℓ', we can smoothly extend g_ℓ to $U_{\ell'}$. Assuming that one can always extend a local solution across transversals as described above, proceeding recursively until no separatrices are left we end up with a global solution to (1.1).

We are going to use the gap to provide a quantitative criterion for the existence of continuous solutions. While the gap of a function clearly depends on the CST chosen, whether it exists and is bounded does not:

Proposition 4.1.1. *If the gap of $g \in C^\infty(\mathbb{R}^2)$ between two adjacent separatrices s_1 and s_2 with respect to a CST $\mathcal{T}_\xi$ exists and it is finite, then it exists and it is finite also with respect to every other CST $\mathcal{T}'_\xi$.*

Proof. Let $\ell'_1, \ell'_2 \in \mathcal{T}'_\xi$ be the two transversal to s_1 and s_2 in the second CST. Then

$$\underset{\mathcal{T}'_\xi}{gap}(g; s_1, s_2) = \underset{\mathcal{T}_\xi}{gap}(g; s_1, s_2) + A_1 + A_2$$

for

$$A_1 = \int_{p'_1}^{p_1} g\left(\Phi^t_\xi(p'_1)\right) dt\,,\ A_2 = \int_{p_2}^{p'_2} g\left(\Phi^t_\xi(p_2)\right) dt$$

where the integral defining A_i, $i = 1, 2$, is evaluated along s_i. Recall that, due to the method of characteristics, the values on a leaf of a local solution to the cohomological equation are completely determined by the value of the solution in any point of the leaf and they are finite on the whole leaf iff they are finite at a single point. Hence, if the gap of g between s_1 and s_2 with respect to $\mathcal{T}_\xi$, both A_i are finite since they are given by integrals of bounded functions over compact sets. □

It is already implicit in the previous proof that the existence and boundedness of the gap of a function g is related to the extendability of local solutions of the cohomological equation having g as rhs. Below we prove this fact and then use it to prove the main result of the section.

Proposition 4.1.2. *A global continuous solution to $L_\xi f = g$ exists iff g has finite gap between every pair of adjacent separatrices of $\mathcal{F}_\xi$.*

Proof. We point out first that a continuous solution to $L_\xi f = g$, $g \in C^\infty(\mathbb{R}^2)$, is much more regular than it sounds since all such solutions are, by definition, smooth in the ξ direction. In particular the integral of df along the integral trajectories of ξ is well-defined even for continuous solutions of (1.1) since the restriction of df on these integral trajectories depends only on $L_\xi f$.

The condition in the hypothesis of the theorem is clearly necessary for, if a continuous solution f exists, then for a given $\mathcal{T}_\xi$ we have

$$\underset{\mathcal{T}_\xi}{gap}(g; s_1, s_2) = \lim_{p \to p_1} \int_0^{T_p} df = f(p_2) - f(p_1)\,.$$

Note that the gap of g between s_1 and s_2 depends only on the intersection of the two separatrices with the relative transversals in $\mathcal{T}_\xi$.

Now assume that a solution f_1 is defined in $U_1 = \pi^{-1}(\ell_1)$ and that the gap of g between s_1 and s_2 is finite. Then the restriction of f_1 on ℓ_2 can be continued to a continuous function on the whole ℓ_2 and therefore, via the the method of characteristics, to the whole $U_2 = \pi^{-1}(\ell_2)$. The new function f_2 defined on $U_1 \cup U_2$ coincides, by construction, with f_1 in $U_1 \cap U_2$, is continuous in $U_1 \cup U_2$ and clearly does not dependent on the choice of the particular CST used in the extension. By proceeding recursively until all separatrices are taken into account we end up with a global continuous solution to (1.1). □

We are now in condition to prove the main theorem of the section:

Theorem 4.1.3. *If ξ has at least a pair of separatrices then* $\dim \operatorname{coker} L_\xi = \infty$.

Proof. We can assume without loss of generality that ξ is complete[3]. Under this assumption the gap of every non-zero constant function is infinite for it is proportional to T_p, which clearly diverges for $p \to p_1$. Then the gap diverges also on every function which is minored by a non-zero constant, e.g. the polynomials $p_{n,m}(x, y) = 1+x^{2n}+y^{2m}$, so that the image of L_ξ misses infinitely many linearly independent functions, i.e. $\dim \operatorname{coker} L_\xi = \infty$. □

Remark 4.1.4. *Observe that, in particular, Theorem 4.1.3 shows that there there is a qualitative difference between the case of a single PDO L_ξ acting on $C^\infty(M)$ and the case of an n-ple $\mathcal{L}_1 = (L_{\xi_1}, \dots, L_{\xi_q})$, $q \geq 2$, acting on $(C^\infty(M))^q$. Indeed in the latter case, as shown in Section 2.4.2, for a generic choice of the ξ_i the operator $\mathcal{L}_1$ is always surjective.*

4.2 $L_\xi f > 0$

Finding criteria to characterize functions belonging to the image of L_ξ is hard and in the case of a generic regular vector field we cannot state much more than the fact that a necessary condition (but far from being sufficient) to belong to it is to have finite gap between all pairs of adjacent separatrices. More can be said for the vector fields which are transversally Hamiltonian, which makes crucial studying the solvability of the differential inequality $L_\xi f > 0$.

[3]This is true for any smooth vector field on a manifold, e.g. see [34], Proposition 1.19; in this case, since ξ is regular, we could simply assume that it has unitary Euclidean length.

Proposition 4.2.1. *Let $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$, $\Omega_0 = dx \wedge dy$ and $\omega_\xi = i_\xi \Omega_0$. The following conditions are equivalent:*

1. *$\mathcal{F}_\xi$ is transversally Hamiltonian;*
2. *the inequality $L_\xi f > 0$ has a smooth solution;*
3. *$\omega_\xi \wedge df$ is a volume form for some $f \in C^\infty(\mathbb{R}^2)$.*

Proof. Let $\mathcal{G}$ be a Hamiltonian foliation transversal to $\mathcal{F}_\xi$ and G a Hamiltonian for $\mathcal{G}$. Since $T\mathcal{G} = \ker dG$ we must have $dG(\xi) \neq 0$ at every point, so that either $L_\xi G > 0$ or $L_\xi(-G) > 0$ and viceversa. Part 3 is due to the fact that $\omega_\xi \wedge dG = i_\xi dG\, \Omega_0 = L_\xi G\, \Omega_0$. □

Now we generalize, as mentioned in Section 3.2.1, Weiner's Lemma 3.2.1 to all finite type vector fields. We start with a preparatory Lemma:

Lemma 4.2.2. *Let ξ be a regular vector field of finite type. Then $\mathcal{F}_\xi$ admits a CST with the following property: for each separatrix $s \in \mathcal{S}$, the saturated open set $\pi_\xi^{-1}(\pi_\xi(\ell))$ of all leaves cutting the corresponding transversal $\ell \in \mathcal{T}$ is equal to the union of s with the two canonical regions having s as boundary component.*

Proof. Let s be a separatrix, U one of the two canonical regions having s as boundary, ℓ the corresponding transversal in $\mathcal{T}$ and ℓ_U the connected component of $\ell \setminus s$ which intersects U. Since U admits a global transversal, there is a natural diffeomorphism ψ of U into $\mathbb{R}$ sending the leaves of $\mathcal{F}_\xi$ into vertical lines. If $\pi_\xi^{-1}(\pi_\xi(\ell_U)) \neq U$ there is no geometrical obstruction to make $\psi(\ell_U)$ either shorter or longer in the horizontal direction while keeping it transversal to the vertical direction and without modifying it close to s so that the first projection of $\psi(\ell)$ on the first factor is surjective. After we do the same on the second canonical region V we are left with a new transversal ℓ' such that $\pi_\xi^{-1}(\pi_\xi(\ell')) = U \cup V \cup s$. □

Theorem 4.2.3. *Every regular vector field of finite type is transversally Hamiltonian.*

Proof. We can assume without loss of generality that ξ is complete and denote by $\mathcal{T}_\xi$ any CST having the property described in the Lemma above. The collection of open subsets $V_{s,i}$ defined by

$$V_{s,i} = \{\Phi^t_\xi(\ell_s) \,|\, t \in (i, i+1)\}\,,\ s \in \mathcal{S}_\xi\,,\ i \in \mathbb{Z}\,,$$

where Φ_ξ is the flow of ξ and ℓ_s the transversal associated to s in $\mathcal{T}_\xi$, is a locally finite open cover of $\mathbb{R}^2$. Indeed by hypothesis the union of the $\pi_\xi(\ell_i)$ covers $\mathcal{F}_\xi$ and therefore under the flow Φ_ξ the ℓ_i cover the whole plane. Moreover, since the boundary of every canonical region has only finitely many components, only finitely many of the $V_{s,i}$ cover any given point.

Inside each $V_{s,i}$ every point p can be written as $\Phi^t_\xi(q)$ for some $q \in \ell_s$ so that we can define a smooth function $f_{s,i}(\Phi^t_\xi(q)) = \phi(t)$, where ϕ is any smooth function strictly monotonic for $t \in (0,1)$ and such that $\phi|_{(-\infty,0)} \equiv 0$ and $\phi|_{(1,\infty)} \equiv 1$. Since each $V_{s,i}$ divides the plane in two connected components, each $f_{s,i}$ can be extended to a smooth function on the whole plane by setting it identically to 1 in the component containing $\Phi^1_\xi(\ell_s)$ and identically 0 in the other. A direct calculation shows that $L_\xi f_{s,i}(p) = \phi'(t) > 0$ within each $V_{s,i}$ while $L_\xi f_{s,i}$ is identically zero outside of it. Now recall that the set $\mathcal{S}_\xi \times \mathbb{Z}$ is countable and let $n_{s,i}$ be any bijection of it with $\mathbb{N}$. The series

$$f = \sum_{s\in\mathcal{S}_\xi, i\in\mathbb{Z}} 2^{-n_{s,i}} f_{s,i}$$

converges to a continuous function (because the $f_{s,i}$ are uniformly bounded) which is actually smooth because the derivatives of all positive orders of the $f_{s,i}$ have compact support. By construction $L_\xi f \geq 0$ but the inequality is strict because for every x_0 there exists at least one index (s_0, i_0) such that $L_\xi f_{s_0,i_0} > 0$. □

Note that the inequality $L_\xi f > \epsilon$, with $\epsilon > 0$, requires stricter conditions to be solvable no matter how small ϵ is. E.g. it admits no smooth solutions if ξ is complete for in that case, as pointed out in the previous section, all gaps of the constant function ϵ (and, a fortiori, all gaps of every function not smaller than it) would be infinite.

4.3 $L_\xi(C^\infty(\mathbb{R}^2))$

From this point on we will assume that ξ is transversally Hamiltonian and we will denote by $F \in C^\infty(\mathbb{R}^2)$ a generator of $\ker L_\xi$, so that $\ker L_\xi = F^*\left(C^\infty(\mathbb{R})\right)$, by $\mathcal{G}$ the Hamiltonian foliation transversal to $\mathcal{F}_\xi$ and by G any Hamiltonian of $\mathcal{G}$.

A fundamental tool in our analysis will be the map $\Phi_{FG} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $x' = F(x,y)$, $y' = G(x,y)$. Assume first that ξ is intrinsically Hamiltonian, so that F is regular. In this case Φ_{FG} is an immersion, since also G is regular and the level sets of F and G are everywhere transversal by hypothesis, so that it induces on the source space the following metric and symplectic structures:

$$g_{FG} = \Phi^*_{FG}((dx')^2 + (dy')^2) = (dF)^2 + (dG)^2\,,\ \Omega_{FG} = \Phi^*_{FG}(dx' \wedge dy') = dF \wedge dG\,.$$

In particular Φ_{FG} induces on the source space complex structure J_{FG}, whose real and imaginary spaces are $T\mathcal{F}_\xi$ and $T\mathcal{G}$, and a Poisson structure $\{,\}_{FG}$. Via Ω_{FG} we can build a pair of commuting regular vector fields respectively tangent to $\mathcal{F}_\xi$ and $\mathcal{G}$. Recall that the Hamiltonian vector field η associated to a Hamiltonian H with respect to a symplectic form Ω is defined by the relation $i_\eta\Omega = dH$; in this case we write, with a slight abuse of notation, that $\eta = \Omega^{-1}(dH)$.

Proposition 4.3.1. *Let* $\xi'_F = -\Omega^{-1}_{FG}(dF)$, $\xi_F = -\Omega_0^{-1}(dF)$, $\xi'_G = \Omega^{-1}_{FG}(dG)$ *and* $\xi_G = \Omega_0^{-1}(dG)$. *The following relations hold:*

1. $\Omega_{FG} = (L_{\xi_F} G)\,\Omega_0$.
2. $\xi'_F = \frac{1}{L_{\xi_F}G}\xi_F\,,\ \xi'_G = \frac{1}{L_{\xi_F}G}\xi_G$.
3. $L_{\xi'_F}F = 0\,,\ L_{\xi'_F}G = 1\,,\ L_{\xi'_G}F = 1\,,\ L_{\xi'_G}G = 0$.
4. $(\Phi_{FG})_*(\xi'_F) = \partial_{y'}$ *and* $(\Phi_{FG})_*(\xi'_G) = \partial_{x'}$ *within* $\Phi_{FG}(\mathbb{R}^2)$.
5. $\{F, G\}_{FG} = L_{\xi_F}G = 1$.
6. $[\xi'_F, \xi'_G] = 0$.
7. *The pair* (ξ'_F, ξ'_G) *is an orthonormal base for* g_{FG}.
8. $L_{\xi'}g_{FG} = L_{\eta'}g_{FG} = 0$.
9. $L_{\xi'}\Omega_{FG} = L_{\eta'}\Omega_{FG} = 0$.
10. $J_{FG}\xi'_F = \xi'_G\,,\ J_{FG}\xi'_G = -\xi'_F$.

Proof. 1. A direct calculation show that $\xi_F = -\partial_y F\partial_x + \partial_x F\partial_y$, so that $dF \wedge dG = (\partial_x F\partial_y G - \partial_y F\partial_x G)dx \wedge dy = (L_{\xi_F}G)\Omega_0$.

2. It is a direct consequence of the definition of ξ'_F and ξ'_G and (1).

3. It is a direct consequence of (2).

4. Since Φ_{FG} is not an injection, the push-forward of a vector field $(\Phi_{FG})_*(\zeta) = T\Phi_{FG} \circ \zeta \circ \Phi^{-1}_{FG}$ is not well-defined unless $T\Phi_{FG}(\zeta)$ takes the same value on all points of $\Phi^{-1}_{FG}(p)$ for every $p \in \Phi_{FG}(\mathbb{R}^2)$. This is the case for ξ'_F and ξ'_G since we get in both cases a constant vector field:

$$((\Phi_{FG})_*(\xi'_F))(x') = \xi'_F(\Phi^*_{FG}(x')) = \xi'_F(F) = 0$$

$$((\Phi_{FG})_*(\xi'_F))(y') = \xi'_F(\Phi^*_{FG}(y')) = \xi'_F(G) = 1$$

and similarly for ξ'_G.

5. $\{F, G\}_{FG} = \{\Phi^*_{FG}x', \Phi^*_{FG}y'\}_{FG} = \Phi^*_{FG}\{x', y'\}_0 = \Phi^*_{FG}1 = 1$.

6. $[\xi'_F, \xi'_G] = [-\Omega^{-1}_{FG}(dF), \Omega^{-1}_{FG}(dG)] = \Omega^{-1}_{FG}(\{F, G\}_{FG}) = \Omega^{-1}_{FG}(1) = 0$.

7. $g_{FG}(\xi'_F, \xi'_F) = \left(dF(\xi'_F)\right)^2 + \left(dG(\xi'_F)\right)^2 = (L_{\xi'_F}F)^2 + (L_{\xi'_F}G)^2 = 0 + 1$ and similarly for the other combinations.

8. It is a direct consequence of the previous item.

9. This just restates that ξ'_F and ξ'_G are Hamiltonian with respect to Ω_{FG}.

10. It is due to the fact that both g_{FG} and Ω_{FG} are in canonical form with respect to ξ'_F and ξ'_G.

□

When ξ is not intrinsically Hamiltonian but is of finite type then F is not globally regular but nevertheless its differential goes to zero only on some of the separatrices, so that the restriction of Φ_{FG} to each of the canonical regions of ξ is still an immersion. Correspondingly, the pair of commuting regular vector fields ξ'_F and ξ'_G is well defined within the canonical regions but, while ξ'_F is globally well-defined, ξ'_G diverges on the separatrices where dF is zero. Note that there is no way to find a global substitute for ξ'_G:

Proposition 4.3.2. *Let $\mathcal{F}$ be a plane foliation of finite type. Then a pair of commuting regular linearly independent vector fields (ξ, η), with ξ tangent to $\mathcal{F}$, exists iff $\mathcal{F}$ is Hamiltonian.*

Proof. We showed in previous proposition that such pair always exists if $\mathcal{F}$ is Hamiltonian. Assume then that it is not. In this case we can always find a smooth function F with no maxima or minima whose differential vanishes on some of the separatrices and whose level sets are the leaves of $\mathcal{F}$ and a second function G, this one regular on the whole plane, whose level sets are always transversal to $\mathcal{F}$. Correspondingly we can always find two vector fields ξ and η s.t.

$$L_\xi F = 0\,,\; L_\xi G = 1\,,\; L_\eta G = 0\,,\; L_\eta F \geq 0\,.$$

Let now α e β the two smooth functions s.t. $[\xi, \eta] = \alpha\xi + \beta\eta$. Then

$$\alpha = \alpha L_\xi G + \beta L_\eta G = L_{[\xi,\eta]} G = L_\xi(L_\eta G) - L_\eta(L_\xi G) = 0$$

while

$$\beta L_\eta F = \alpha L_\xi F + \beta L_\eta F = L_{[\xi,\eta]} F = L_\xi(L_\eta F) - L_\eta(L_\xi F) = L_\xi(L_\eta F)$$

namely $\beta = L_\xi[\log L_\eta F]$. Since $[\xi, \eta]$ has only the η component, the only thing we can do to make the commutator vanish is multiplying η by some non-zero factor μ since any other change would just introduce a ξ component. On the other side

$$[\xi, \mu\,\eta] = L_\xi \mu\,\eta + \mu[\xi, \eta] = L_\xi \mu\,\eta + \mu\,\beta\,\eta$$

leading to $\mu = 1/L_\eta F$; this function though is not smooth because the differential of F vanishes on some of the separatrices. $\square$

Let us turn now to the study of the image of L_ξ. This is clearly equivalent to studying the image of $L_{\xi'_F}$ but the latter is more convenient for the following two propositions:

Proposition 4.3.3. *The cohomological equation $L_{\xi'_F} f(x, y) = g(x, y)$, restricted to the subalgebra $\Phi^*_{FG}\left(C^\infty(\mathbb{R}^2)\right) = \{\Phi^*_{FG}\hat{f} \,|\, \hat{f} \in C^\infty(\mathbb{R}^2)\}$, writes, in the image of Φ_{FG}, as*

$$\frac{\partial}{\partial y'} \hat{f}(x', y') = \hat{g}(x', y') \tag{4.1}$$

where $\hat{f} = (\Phi_{FG})_* f$ and $\hat{g} = (\Phi_{FG})_* g$.

Proof. In general Φ_{FG} is not injective so that, while the pull-back of function $\Phi^*_{FG}\hat{f} := \hat{f}\circ\Phi_{FG}$ is well-defined on the whole $C^\infty(\mathbb{R}^2)$, the push forward $(\Phi_{FG})_* f := f\circ\Phi^{-1}_{FG}$ leads to a well-defined function only within the subalgebra $\Phi^*_{FG}\left(C^\infty(\mathbb{R}^2)\right)$. Then from point (3) of Proposition 4.3.1 follows that

$$(\Phi_{FG})_*\left(L_{\xi'_F}\left(\Phi^*_{FG}\hat{f}\right)\right) = L_{(\Phi_{FG})_*\xi'_F}\left((\Phi_{FG})_*\Phi^*_{FG}\hat{f}\right) = \frac{\partial}{\partial y'}\hat{f}$$

□

Theorem 4.3.4. *A function $g\in C^\infty(\mathbb{R}^2)$ belongs to $L_{\xi'_F}(C^\infty(\mathbb{R}^2))$ iff all functions $L^k_{\xi'_G} g$, $k\in\mathbb{N}$, have finite gap for all pairs of adjacent separatrices of ξ'_F.*

Proof. As we already pointed out, every continuous solution to $L_{\xi'_F} f = g$ is automatically smooth in the ξ'_F direction, i.e. $L^k_{\xi'_F} f$ is continuous for every $k\in\mathbb{N}$.

Assume first that ξ'_F is intrinsically Hamiltonian. Since ξ'_F and ξ'_G commute and are globally well-defined, the first derivative in the ξ'_G direction satisfies the cohomological equation $L_{\xi'_F}\left(L_{\xi'_G} f\right) = L_{\xi'_G} g$ and analogously the k-th derivative in the ξ'_G direction satisfies $L_{\xi'_F}\left(L^k_{\xi'_G} f\right) = L^k_{\xi'_G} g$. Now we can use the claim of Lemma 4.1.2 to conclude that each $L^k_{\xi'_G} f$ is globally continuous iff $L^k_{\xi'_G} g$ has finite gap between every pair of adjacent separatrices.

Assume now that ξ'_F is of finite type, so that ξ'_G is only well-defined within the canonical regions of ξ'_F. By repeating the same kind of arguments used in Lemma 4.1.2 it is clear that we can extend a smooth solution within a saturated open set to the whole plane iff the gap of $L^k_{\xi'_G} g$ has finite gap between every pair of adjacent separatrices. Note indeed that in the definition of gap the values of ξ'_G on the separatrices are never used so the fact that ξ'_G diverges on some of them does not hinder the evaluation of the gap. □

From Proposition 4.3.3 and the surjectivity of $\partial_{y'}$ we get a large explicit subalgebra of the image of $L_{\xi'_F}$:

Proposition 4.3.5. $\Phi^*_{FG}\left(C^\infty(\mathbb{R}^2)\right)\subset L_{\xi'_F}(C^\infty(\mathbb{R}^2))$

This fact corresponds to two elementary observations: one, algebraic, that

$$L_{\xi'_F}\hat{f}(F,G) = L_{\xi'_F}F\,\partial_{x'}\hat{f}(F,G) + L_{\xi'_F}G\,\partial_{y'}\hat{f}(F,G) = \partial_{y'}\hat{f}(F,G)\,;$$

the other, geometric, that the constant vertical vector field $\partial_{y'}$ on $\Phi_{FG}(\mathbb{R}^2)$ can always be extended to the whole plane, where it is surjective on $C^\infty(\mathbb{R}^2)$.

4.4 Local behaviour of functions of $L_\xi(C^\infty(\mathbb{R}^2))$ close to a pair of adjacent separatrices

Proposition 4.3.3 shows that locally, in the image of the map Φ_{FG}, the cohomological equations relative to vector fields ξ'_F look all the same, independently on the topology of their leaf spaces; the qualitative difference between them resides rather in the global geometry of the map Φ_{FG}. It is easy to verify that, as soon as ξ'_F has at least two pairs of separatrices, Φ_{FG} cannot be injective, which is not optimal for several reasons. We bypass this problem by considering the map $\hat\Phi_{FG} : \mathbb{R}^2 \to \mathbb{R}^4$ defined by $\hat\Phi_{FG}(x,y) = (x, y, F(x,y), G(x,y))$. By construction $\hat\Phi_{FG}$ is a diffeomorphism between $\mathbb{R}^2$ and $\Gamma_{FG} = \hat\Phi_{FG}(\mathbb{R}^2) \subset \mathbb{R}^4$, the graph of Φ_{FG}. The symplectic, metric and almost complex structures determined on $\mathbb{R}^2$ by F and G, as pointed out at the beginning of the previous section, induce the same structures on Γ_{FG} via the push-forward $(\hat\Phi_{FG})_*$. We use on $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ coordinates (x, y, x', y') and denote by π_1 and π_2 the projections on the first and second factor. By definition $\pi_1 \circ \hat\Phi_{FG} = \mathrm{id}_{\mathbb{R}^2}$ and $\pi_2 \circ \hat\Phi_{FG} = \Phi_{FG}$, so Γ_{FG} admits (x, y) as global coordinates and (F, G) as local coordinates at every point. A direct calculation shows that

$$(\hat\Phi_{FG})_*(\xi'_F) = \xi'_F \oplus \partial_{y'}\ ,\ (\hat\Phi_{FG})_*(\xi'_G) = \xi'_G \oplus \partial_{x'}\ .$$

In particular the projection on the second factor of the image of the leaves of $\mathcal{F}_\xi$ and $\mathcal{G}$ are, respectively, vertical and horizontal straight lines in the plane (x', y'). All leaves which are inseparable one from the other are mapped to disjoint open intervals of the same line, so that the images in the graph of any pair of adjacent separatrices of ξ'_F are separated by a vertical closed bounded interval I.

Proposition 4.4.1. *For every pair of separatrices s_1 and s_2 of ξ'_F, with $a = F|_{s_1\cup s_2}$, there exists a saturated open neighbourhood U of s_1 and s_2 on which Φ_{FG} is injective and $\Phi_{FG}(U \cap \Phi^{-1}_{FG}((a_1, a_2) \times (c_1, c_2))) = (a_1, a_2) \times (c_1, c_2) \setminus R$, where $R = [a, a_2) \times [b_1, b_2]$ or $R = (a_1, a] \times [b_1, b_2]$, both a_i and c_i can be infinite and $c_1 < b_1 \le b_2 < c_2$.*

Proof. Let $p_i \in s_i$, $i = 1, 2$, be any two points on the two separatrices, set $c_i = G(p_i)$ and denote by ℓ_i be the two leaves of $\mathcal{G}$ passing through the p_i. The two numbers c_1 and c_2 cannot be equal since the restriction of G to any leaf of $\mathcal{F}_{\xi'_F}$ is strictly monotonic and, because of the inseparability of s_1 and s_2, there are leaves of $\mathcal{F}_{\xi'_F}$ cutting both ℓ_1 and ℓ_2; in particular $G(s_1)\cap G(s_2) = \emptyset$. Assume that $c_1 < c_2$ (otherwise switch the names of the points), set $U_i = \pi_\xi^{-1}(\ell_i)$, $i = 1, 2$ and denote by V and Λ respectively the union and intersection of U_1 and U_2.

Assume first that Λ is contained in $F < a$. We claim that the restriction of Φ_{FG} to V is injective. Indeed let $A_i = U_i \setminus \Lambda$, $i = 1, 2$, so that $V = \Lambda \sqcup A_1 \sqcup A_2$. Clearly $\Phi_{FG}|_\Lambda$ is injective since Λ fibers on $\ell_1 \cap \Lambda$, each fiber being a leaf of cF_ξ, with G strictly monotonic on each fiber and F strictly monotonic on the base.

Moreover, $F(\Lambda) \subset (-\infty, a)$ by assumption. Similarly, each A_i fibers on $\ell_i \cap A_i$ so that $\Phi|_{A_i}$ is injective too; this time though $F(A_i) \subset [a, \infty)$ and, moreover, $G(A_i) = G(s_i)$. Consider now the set $V' = V \cap G^{-1}((c_1, c_2))$ and let s be any leaf of $\mathcal{F}_\xi$ inside Λ. The sets of leaves of $\mathcal{G}|_{V'}$ intersecting, respectively, s_1 and s_2 cut s in two disjoint open intervals (c_1, b_1) and (b_2, c_2); in particular all leaves of $\mathcal{G}|_{V'}$ corresponding to the values in the closed interval $[b_1, b_2]$ do not intersect neither s_1 nor s_2 and are such that s_1 and s_2 lie on different components with respect to each of them. Finally, let $F(\ell_1) = (a_1, a_2')$ and $F(\ell_2) = (a_1, a_2'')$. Then $\Phi_{FG}(\Lambda \cap G^{-1}((c_1, c_2))) = (a_1, a) \times (c_1, c_2)$, $\Phi_{FG}(A_1 \cap G^{-1}((c_1, c_2))) = [a, a_2') \times (c_1, b_1)$ and $\Phi_{FG}(A_2 \cap G^{-1}((c_1, c_2))) = [a, a_2'') \times (b_2, c_2)$ so that $\Phi_{FG}(V \cap F^{-1}((a_1, a_2)) \cap G^{-1}((c_1, c_2))) = (a_1, a_2) \times (c_1, c_2) \setminus R$ for $a_2 = \min\{a_2', a_2''\}$.

In case V is contained in $F > a$, we use the chart $\tilde{\Phi}_{FG} = (-F, G)$ and repeat the argument above. $\square$

We call the chart $(U \cap \Phi_{FG}^{-1}((a_1, a_2) \times (c_1, c_2)), \Phi_{FG})$ [4] a *normal chart for s_1 and s_2*. By Proposition 4.3.4 there are countably many conditions that must be satisfied for each one of the intervals between pairs of adjacent separatrices so that equation (4.1) admits a smooth solution. Since in Γ_{FG} there is a natural family of transversals for $\mathcal{F}_\xi$ these conditions can be restated more properly for this setting in the following way. Let $I = \{a\} \times [b_1, b_2]$ the vertical interval separating a pair of adjacent separatrices s_1 and s_2 in a normal chart. Every such interval determines a rings homomorphism $\theta_I^{(r)} : SG_I^r \to SG_a^r$ defined as follows. Given $\mathfrak{g} \in SG_I^r$, let $\hat{g} \in \mathfrak{g}$ and $\delta = \min\{c_2 - b_2, b_1 - c_1\}$, choose an arbitrary $\epsilon \in (0, \delta)$ and set $h_I(x') = \int_{b_1-\epsilon}^{b_2+\epsilon} \hat{g}(x', y')dy'$ for $x' \in (a_1, a)$; we define $\theta_I^{(r)}(\mathfrak{g}) = [h_I]_{SG_a^r}$.

Proposition 4.4.2. *The left singular germ of h_I, modulo germs of smooth functions, does not depend on the particular choice of $\epsilon \in (0, \delta)$ and $\hat{g} \in \mathfrak{g}$.*

Proof. Let $h_I'(x') = \int_{b_1-\epsilon'}^{b_2+\epsilon'} \hat{g}'(x', y')dy'$ for $\epsilon' \in (0, \delta)$ and $\hat{g}' \in \mathfrak{g}$. Then the function

$$h_I'(x') - h_I(x') = \int_{b_1-\epsilon'}^{b_2-\epsilon} (\hat{g}'(x', y') - \hat{g}(x', y'))\, dy' +$$

$$+ \int_{b_2+\epsilon}^{b_2+\epsilon'} \hat{g}(x', y')dy' + \int_{b_1-\epsilon'}^{b_1-\epsilon'} \hat{g}(x', y')dy'$$

is smooth in $(a_1, a]$ since the integrands are all smooth in R_I, the last two because g is smooth in $R_I \setminus I$ and the integral intervals lie inside that set for every $x \in (a_1, a]$ and the first because by hypothesis $\hat{g}' - \hat{g}$ is identically zero in some left neighbourhood of I. Adding to g and $\hat{g}$ any function smooth in the whole R_I changes the rhs just by a smooth function. $\square$

[4] Replace Φ_{FG} with $\tilde{\Phi}_{FG}$ if, in the terminology of Proposition 4.4.1, V is contained in $F > a$.

The maps $\theta^{(r)}_I$ then are well-defined. It is clear from the definition of h_I that $\theta^{(r)}_I$ is a $C^r_x(\mathbb{R})$-module homomorphism, where $C^r_x(\mathbb{R})$ is the algebra of C^r functions depending on x' only, since

$$\int_{b_1-\epsilon}^{b_2+\epsilon} f(x')\hat{g}(x',y')dy' = f(x')\int_{b_1-\epsilon}^{b_2+\epsilon} \hat{g}(x',y')dy'\,,$$

and commutes with the derivatives with respect to x', i.e. $\theta^{(r)}_I(\partial^k_{x'}\hat{g}) = \partial^k_{x'}\theta^{(r)}_I(\hat{g})$.

Next proposition shows that the maps $\theta^{(r)}_I$ determine the solvability of the cohomological equation.

Theorem 4.4.3. *Let $\{I_j\}$ be the set of all (vertical, closed) intervals between adjacent separatrices in Γ_{FG} and $\theta^{(r)}_{I_j}$ the corresponding ring homomorphisms. Then $g \in L_{\xi'_F}(C^r(\mathbb{R}^2))$ iff $[(\hat{\Phi}_{FG})_* g]_{SG^r_{I_j}} \in \ker\theta^{(r)}_{I_j}$ for all $\theta^{(r)}_{I_j}$.*

Proof. Let $I = \{a\}\times[b_1,b_2]$ be the vertical interval which separates two adjacent separatrices of ξ'_F in a normal chart for the corresponding adjacent separatrices s_1 and s_2 and set $\hat{g} = (\Phi_{FG})_* g$ within that chart. Then

$$\lim_{x'\to a^-}\int_{b_1-\epsilon}^{b_2+\epsilon} \partial^k_{x'}\hat{g}(x',y')dy'$$

is exactly the gap of $\Phi^*_{FG} g$ between s_1 and s_2 with respect to the pair of transversals which are the counterimages of $y' = b_1-\epsilon$ and $y' = b_2+\epsilon$ and the gap exists and is finite if and only if those functions can all be extended to continuous functions for all $k < r$, which in turn means that the (germ of the) function $\int_{b_1-\epsilon}^{b_2+\epsilon}\partial^k_{x'}\hat{g}(x',y')dy'$ can be extended to a smooth map up to $x' = a$, i.e. $[(\hat{\Phi}_{FG})_* g]_{SG^r_I} \in \ker\theta^{(r)}_I$. Now the claim follows immediately from Theorem 4.3.4. $\square$

The $C^r(\mathbb{R})$-modules $\Theta^r_{I_j} = \ker\theta^{(r)}_{I_j}$ contain therefore the (left singular) germs of all functions for which the cohomological equation is solvable in the neighbourhood of a pair of adjacent separatrices. Modulo isomorphisms there are only two such spaces: the one relative to $J = \{0\}\times[-1,1]$ and the one relative to $O = \{(0,0)\}$; moreover $\Theta^r_O \subset \Theta^r_J$.

Proposition 4.4.4. *The spaces Θ^r_O satisfy the following properties:*

1. *Θ^r_O contains the singular left germs of all y'-odd*[5] *C^r functions;*
2. *Θ^r_O contains the singular left germs of some but not all y'-even C^r functions;*
3. *Θ^{r+1}_O is strictly contained in Θ^r_O.*

[5] We say that $f(x,y)$ is y-odd if $f(x,-y) = -f(x,y)$ and y-even if $f(x,-y) = f(x,y)$.

Proof. 1. If $\hat{g}$ is y'-odd then also every $\partial^k_{x'}\hat{g}$ is so for every $k \leq r$; then $\int_{-\epsilon}^{\epsilon} \partial^k_{x'}\hat{g}(x',y')dy'$ is identically zero for every $k \leq r$ and therefore it can be extended smoothly to a C^r function up to $x'=0$.

2. Consider $\hat{g}(x',y') = e^{-(y')^2/(x')^2}/\sqrt{-\pi x'} \in C^\infty(\mathbb{R}^2 \setminus (0,0))$, so that

$$\lim_{x\to 0^-} g(x',y') = 0\,,\ y' \neq 0\,;\ \lim_{x\to 0^-} g(x',0) = \infty\,;\ \int_{-\infty}^{\infty} g(x',y')dy' = 1\,,\ \forall x' \in \mathbb{R}\,.$$

By reparametrizing the y' coordinate we can find a $\hat{g}'$ with the same limits with respect to $x' \to 0$ but such that $\int_{-\epsilon}^{\epsilon} \hat{g}'(x',y')dy' = 1$. Since the $\theta^{(r)}_O$ are homomorphisms of $C^r_x(\mathbb{R})$-modules we can get in this way every C^r function $f(x')$ just by multiplying $g'(x',y')$ by $f(x')$. On the other side, germs of functions diverging too fast, e.g. as $\hat{g}(x',y') = (x')^{-2} + (y')^{-2}$, do not belong to any Θ^r_O.

3. Consider $\hat{g}(x',y') = \dfrac{x'}{\sqrt{(x')^2+(y')^2}} \in C^\infty(\mathbb{R}^2 \setminus (0,0))$. The germ of the corresponding $h_O(x') = 2x' \log\left[2\left(y' + \sqrt{(x')^2+(y')^2}\right)\right]^{\epsilon}_{y'=0}$ can be extended at 0 to a C^0 (but not C^1) function. By integrating r times $\hat{g}$ with respect to x' one can get concrete examples of functions smooth in $\mathbb{R}^2 \setminus (0,0)$ whose germ belongs to Θ^r_O but not to Θ^{r+1}_O. □

An immediate consequence of point (3) of the proposition above is the following:

Corollary 4.4.5. *Let $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$, $L^{(r)}_\xi$ the restriction of L_ξ to $C^r(\mathbb{R}^2)$ and let $C^r_\xi(\mathbb{R}^2)$ be the set of all functions $f \in C^r(\mathbb{R}^2)$ such that $f+g$ is at most C^r for all $g \in \ker L^{(r)}_\xi$. The inclusions*

$$L^{(r+1)}_\xi\left(C^{r+1}_\xi(\mathbb{R}^2)\right) \cap C^\infty(\mathbb{R}^2) \subset L^{(r)}_\xi\left(C^r_\xi(\mathbb{R}^2)\right) \cap C^\infty(\mathbb{R}^2)$$

are strict for every $r \in \mathbb{N}$.

Proof. The fact that $L^{(r+1)}_\xi\left(C^{r+1}(\mathbb{R}^2)\right) \cap C^\infty(\mathbb{R}^2) \subset L^{(r)}_\xi\left(C^r(\mathbb{R}^2)\right) \cap C^\infty(\mathbb{R}^2)$ is trivially true because $L^{(r)}_\xi(f+g) \in C^\infty(\mathbb{R}^2)$ for each $f \in C^\infty(\mathbb{R}^2)$, $g \in \ker L^{(r)}_\xi$. Our claim is that the inclusion is true even when we restrict L_ξ to the space of functions which are "strongly C^r" with respect to ξ, i.e. those that cannot be made smoother by adding to them an element of the kernel of $L^{(r)}_\xi$. Consider indeed the concrete case used in point (3) of Proposition 4.4.4: in a normal chart, where the two separatrices are given by $x'=0$, $y'>a$ and $x'=0$, $y'<a$, the (local) primitive of $\hat{g}(x',y') = x'/\sqrt{(x')^2+(y')^2}$ is $f(x',y') = x'\log\left[2\left(y'+\sqrt{(x')^2+(y')^2}\right)\right]$, which is C^0 but not C^1 because the first derivative with respect to x' diverges on the second separatrix. Since the divergence takes place only on one of the separatrices, there is no way to eliminate it by adding a function belonging to the kernel of L_ξ. □

In the following subsections we work out in detail two model examples.

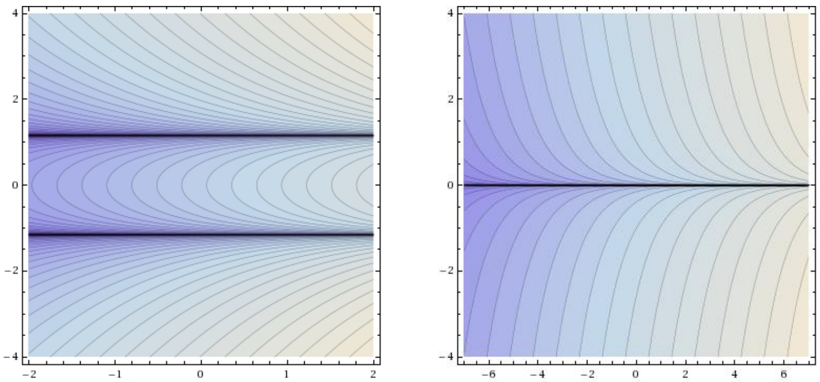

Figure 4.1: Level sets of $F(x,y)=(y^2-1)e^x$ (left) and $G(x,y)=ye^x$ (right). The first foliation has separatrices $y=\pm1$, the second has none.

4.4.1 $\xi_n = (1-n+(1+n)y)\,\partial_x + (1-y^2)\,\partial_y$

The ξ_n, $n\in\mathbb{N}$, are all of finite type since they are polynomial. In particular they all have exactly two separatrices, the straight lines $y=\pm1$, which bound the canonical region $\mathbb{R}\times(-1,1)$. The function $F_n(x,y)=(1+y)^n(1-y)e^x$ is a functional generator for $\ker L_{\xi_n}$ so the only intrinsically Hamiltonian among them is $\xi_1 = 2y\,\partial_x + (1-y^2)\,\partial_y$. All of them are transversal to the same Hamiltonian foliation $\mathcal{G}$ of the level sets of $G(x,y)=ye^x$, which is topologically conjugate with the trivial foliation in parallel straight lines. The 2-form

$$\Omega_{FG} = 2(1+y)^{n-1}(1-(n-1)y+ny^2)e^{2x}\,\Omega_0$$

is degenerate on the separatrix $y=-1$ except in the $n=1$ case, when is globally non-degenerate. Via Ω_{FG} we get

$$\xi'_{F_n} = \frac{1}{2e^x(1-(n-1)y+ny^2)}\xi_n\,,\ \xi'_{G_n} = \frac{1}{2e^x(1+y)^{n-1}(1-(n-1)y+ny^2)}\eta\,,$$

where $\eta = 2\partial_x - 2y\partial_y$. Due to the degeneracy of Ω_{FG}, ξ'_{G_n} diverges on the separatrix $y=-1$ for $n\neq1$.

The image of every Φ_{F_nG} is $\mathbb{R}^2{}_0=\mathbb{R}^2\setminus\{0\}\times[0,\infty)$ and Φ_{F_1G} is an almost complex map between $(\mathbb{R}^2,J_{FG})$ and $(\mathbb{R}^2{}_0,i)$ for

$$J_{FG} = y\,\partial_x\otimes dx + 2\,\partial_x\otimes dy - (1+y^2)/2\,\partial_y\otimes dx - y\,\partial_y\otimes dy\,.$$

The leaves of $\mathcal{F}_{F_n}$ within the canonical region are sent to the vertical lines of the half plane $x<0$ and the separatrices $y=-1$ and $y=+1$ to the half lines $\{0\}\times(-\infty,0)$ and $\{0\}\times(0,+\infty)$ respectively. The leaves lying in the half-plane $y>1$ fill in the vertical half-lines the first quadrant and the ones lying in $y<1$ the fourth quadrant. In this case the maps Φ_{F_nG} are all globally injective. The cohomological equation $L_{\xi_{F_n}}f=g$ maps to

$$\partial_{y'}\hat f(x',y') = \hat g(x',y')\,,\ \hat g\in C^\infty(\mathbb{R}^2{}_0)\,. \tag{4.2}$$

When $\hat{g}$ is smooth on the whole plane clearly (4.2) is always solvable. E.g. all smooth solutions to

$$L_{\xi'_{F_n}} f(x,y) = F_n(x,y)G(x,y) = 2(y^2-1)(y+1)^{n-1} y e^{2x}$$

are given by

$$f(x,y) = \frac{F_n(x,y)G^2(x,y)}{2} + h\left(F_n(x,y)\right) = 2(y^2-1)(y+1)^{n-1} y^2 e^{3x} + h\left(F_n(x,y)\right) ,$$

where $h \in C^\infty(\mathbb{R})$.

In the following we assume $n = 1$ since expressions are much simpler in this case. Consider first the y'-odd function

$$\hat{g}(x',y') = \frac{y'}{\sqrt{(x')^2+(y')^2}} \in C^\infty(\mathbb{R}_0)\,,\ \Phi^*_{FG}\hat{g}(x,y) = \frac{2y}{1+y^2} \in C^\infty(\mathbb{R}^2)\,.$$

By Proposition 4.4.4 the singular left germ of $\hat{g}$ belongs to Θ^∞_O and therefore $g \in L_\xi(C^\infty(\mathbb{R}^2))$. Indeed (4.2) in this case is solved by

$$\hat{f}(x',y') = \sqrt{(x')^2+(y')^2}\,,$$

whose pull-back

$$\Phi^*_{FG} f(x,y) = (1+y^2)e^x$$

is globally smooth. Similarly, $y \in L_{\xi'_F}(C^\infty(\mathbb{R}^2))$ since $y = \Phi^*_{FG}\hat{g}(x,y)$ for the y'-odd singular function $\hat{g}(x',y') = (\sqrt{(x')^2+(y')^2} + x')/y'$.

On the contrary, in case of

$$\hat{g}(x',y') = \frac{x'}{\sqrt{(x')^2+(y')^2}}\ ,\ g(x,y) = \Phi^*_{FG}\hat{g}(x,y) = \frac{1-y^2}{1+y^2}\,,$$

as discussed in Proposition 4.4.4 we have that the germ of $\hat{g}$ belongs to Θ^0_O but not to Θ^1_O; correspondingly all solutions will be C^0 but not C^1. E.g. an explicit solution is given by

$$f(x,y) = \Phi^*_{FG}\left(x' \log\left[2\left(y + \sqrt{(x')^2+(y')^2}\right)\right]\right) = (1-y^2)e^x\left(x + 2\log|1+y|\right)\,.$$

Note that Lie derivatives of f are, as expected, smooth with respect to ξ'_{F_1} direction but singular (on the horizontal straight line $y = -1$) with respect to η. In particular, g belongs to $L_{\xi'_F}(L^1_{loc}(\mathbb{R}^2))$ (where the derivative is intended in the weak sense) but does not belong to any $L_{\xi'_{F_1}}(C^k(\mathbb{R}^2))$, $k > 1$. The same happens in case of $x = \Phi^*_{FG}\hat{g}(x,y)$, where $\hat{g}(x',y') = \log(\sqrt{(x')^2+(y')^2} + x')/2)$. For a thorough discussion about locally integrable solutions of regular polynomial vector fields in the plane depending only on one variable see [5].

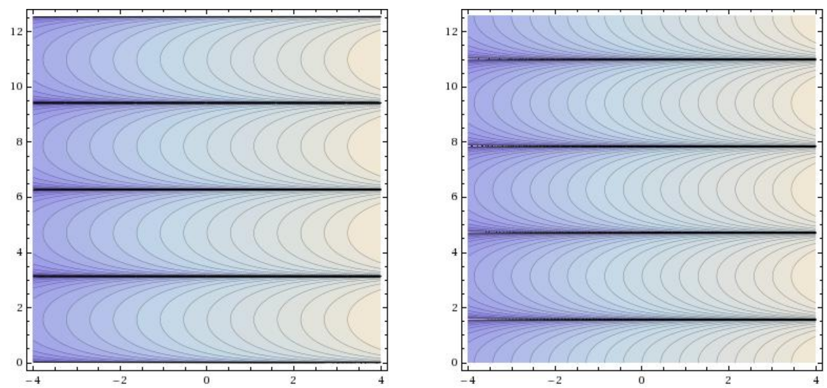

Figure 4.2: Level sets of $F(x,y) = e^x \sin y$ (left) and $G(x,y) = e^x \cos y$ (right). The separatrices of the first foliation are the straight lines $s_k = \{y = k\pi\}$, $k \in \mathbb{Z}$, the ones of the second are the straight lines $s'_k = \{y = \pi/2 + k\pi\}$, $k \in \mathbb{Z}$. Note that $\mathcal{I}_{s_n} = \{s_{n-1}, s_{n+1}\}$, i.e. s_n is inseparable only from s_{n-1} and s_{n+1} (this is possible because the relation of inseparability is not transitive). The same holds for the s'_k.

4.4.2 $\xi_n = (\cos y + (n-1)\cos^2(y/2))\,\partial_x - \sin y\,\partial_y$

The ξ_n, $n \in \mathbb{N}$, are all of finite type for their components are Morse functions depending only on one variable; in this case indeed only the vertical lines can be separatrices and they do not accumulate within any compact set. For every ξ_n the set of separatrices is $\mathcal{S} = \{y = k\pi\,,\ k \in \mathbb{Z}\}$. The function $F_n(x,y) = -\sin^{n-1}(y/2)\sin y\, e^x$ is a functional generator for $\ker L_{\xi_n}$ so that the only intrinsically Hamiltonian among them is $\xi_1 = \cos y\,\partial_x - \sin y\,\partial_y$. A Hamiltonian transversal foliation $\mathcal{G}_n$ for ξ_n is given by the level sets of $G_n(x,y) = \cos y\, e^{x/n}$. The 2-form

$$\Omega_{FG} = [(n-1)(2\cos y - \cos(2y)) + 3n + 1]\sin^{n-1}(y/2)e^{(n+1)x/n}\,\Omega_0/4n$$

is degenerate on the separatrices $y = 2k\pi$, except of course in the $n = 1$ case when is globally non-degenerate. Via Ω_{FG} we get

$$\xi'_{F_n} = \frac{2ne^{-x/n}}{n+1+(n-1)(\sin^2 y - \cos y)}\xi_n,\ \xi'_{G_n} = \frac{-2\sin^{1-n}(y/2)e^{-x}}{n+1+(n-1)(\sin^2 y - \cos y)}\eta\,,$$

where $\eta = n\sin y\,\partial_x + \cos y\,\partial_y$. Due to the degeneracy of Ω_{FG}, ξ'_{G_n} diverges on the separatrices $y = 2k\pi$, $k \in \mathbb{Z}$, for $n \neq 1$.

The image of every Φ_{F_nG} is $\mathbb{R}^2 \setminus \{(0,0)\}$. Note that $\Phi_{F_1G_1}$ is an almost complex map with respect to the almost complex structure

$$J_{F_1G_1} = \partial_y \otimes dx - \partial_x \otimes dy\,,$$

so that $\Phi_{F_1G_1}$ is actually a holomorphic map; in fact, in complex coordinates, $\Phi_{F_1G_1}(z) = e^{z+i\pi/4}$ and its graph is the Riemann surface of the complex logarithm. The graphs of all other $\Phi_{F_nG_n}$ are diffeomorphic to it.

Consider just the case of the coordinate functions x and y. The first is y'-even since $2x = \Phi^*_{FG}\hat{g}(x,y)$ for $\hat{g}(x',y') = \log\left[(x')^2 + (y')^2\right]$. A direct calculation shows that

$$[\theta_n(\hat{g})]\,(x') = 2\int_0^{\epsilon} \log\left((x')^2 + (y')^2\right) dy' = 4x\tan^{-1}(\epsilon/x) + 2\epsilon(\log(\epsilon^2 + x^2) - 2)$$

which can be continued to a smooth function up to $x' = 0$. Hence $\hat{g} \in \Theta^\infty_n$ for all n and, correspondingly, $x \in L_{\xi'_F}(C^\infty(\mathbb{R}^2))$. An explicit solution is given by

$$f(x,y) = \Phi^*_{FG}[2x'\tan^{-1}\frac{y'}{x'} + \log[(x')^2 + (y')^2] - 2y'] = 2\,[(x-1)\cos y - y\sin y]\,e^x\,.$$

The second is y'-odd since $y = \Phi^*_{FG}\hat{g}(x,y)$ for $\hat{g}(x',y') = \tan^{-1}(x'/y')$. Hence even in this case $\hat{g} \in \Theta^\infty_n$ for all n, i.e. $y \in L_{\xi'_F}(C^\infty(\mathbb{R}^2))$. An explicit solution is given by

$$f(x,y) = \Phi^*_{FG}[y'\tan^{-1}\frac{x'}{y'} + \frac{1}{2}\log[(x')^2 + (y')^2]] = -\,[y\cos y + x\sin y]\,e^x\,.$$

5

Weak solutions of the cohomological equation in the plane

In this final chapter we extend our results of the previous chapter by studying the existence of *weak* solutions to the cohomological equation in the plane for some class of smooth regular vector fields. We also investigate the stability of the global solvability for the cohomological equation in weighted Sobolev spaces under perturbation with zero order pseudodifferential operators.

We consider the smooth non-singular real vector field in the plane

$$Lu = p(t)\partial_t u + q(t)\partial_x u = f(t,x), \tag{5.1}$$

i.e., p and q are real-valued smooth functions which have no common zeros.

One assumes that there is an integer $N \geq 2$ and $t_1 < \ldots < t_N$ such that

$$p(t) = 0 \Longleftrightarrow t = t_j,\ j = 1, 2, \ldots, N \tag{5.2}$$

with

$$p'(t_j) \neq 0, \quad j = 1, 2, \ldots, N \tag{5.3}$$

and q admits at most one zero in (t_j, t_{j+1}) for $j = 1, 2, \ldots, N-1$.

Note that the lines $\{t = t_j\}$, $j = 1, \ldots, N$, are characteristics for L. We also suppose that p and q are polynomials. Our results are true under weaker restrictions on p and q, but we prefer to exhibit the main novelties avoiding highly technical arguments and capturing particular cases of L of interest in geometry and dynamical systems (see [50] for foliations and the previous chapter for its action on $C^\infty(\mathbb{R}^2)$). For example,

$$L_0 u = (1-t^2)\partial_t u - 2t\partial_x u$$

and, more generally,

$$L_{\lambda,k} u = (1-t^2)\partial_t u + \lambda t^k \partial_x u \tag{5.4}$$

for $\lambda \neq 0,\ k \in \mathbb{N}$.

The first main goal of the chapter is to show that the existence of separatrix type phenomena for (5.1) is the only obstruction for the surjectivity in $C^\infty(\mathbb{R}^2)$ of L. Moreover, we exhibit functional spaces associated to the separatrix strips where we can solve globally this cohomological equation in $\mathbb{R}^2$ and investigate the stability of this global solvability under perturbations of L with zero order pseudodifferential operators in x.

Definition 5.0.6. *A strip $S_j = \{(t,x) : t \in (t_j, t_{j+1}), x \in \mathbb{R}\}$, with $j \in \{1,\dots,N-1\}$, is a separatrix for the vector field L above if all characteristic curves $x = x(t;\tau,y)$, starting at a point $(\tau,y) \in S_j$ satisfy either*

$$\lim_{t\to t_j^+} x(t;\tau,y) = \lim_{t\to t_{j+1}^-} x(t;\tau,y) = +\infty$$

or

$$\lim_{t\to t_j^+} x(t;\tau,y) = \lim_{t\to t_{j+1}^-} x(t;\tau,y) = -\infty$$

We state the first new result of the chapter:

Theorem 5.0.7. *The following assertions are equivalent:*

i) the vector field L is not surjective in $C^\infty(\mathbb{R}^2)$;

ii) the vector field L admits a separatrix S_j, for some $j \in \{1,\dots,N-1\}$;

iii) there exists $j \in \{1,\dots,N-1\}$ and $\theta_j \in (t_j,t_{j+1})$ such that $q(\theta_j) = 0$ and q has opposite signs in (t_j,θ_j) and (θ_j,t_{j+1}).

In particular, the operators $L_{\lambda,k}$ are not surjective in $C^\infty(\mathbb{R}^2)$ if and only if k is odd.

To illustrate the non-surjectivity for simple example we point out that nonzero constants do not belong to $L_0(C^\infty(\mathbb{R}^2))$. Direct calculations implies that

$$L_0 u = \ c$$

has a weak solution

$$u(t,x) = \frac{c}{2} \ln\left|\frac{1+t}{1-t}\right| .$$

We show for more general classes of rhs $f \in C^\infty(\mathbb{R}^2)$ that every solution has singularity either at $t = 1$ or $t = -1$ (see Section 5.4 for more details).

This example shows that in order to solve globally $Lu = f$ one should allow some (weak) singularities of the type L^1_{loc} near the adjacent characteristics forming the separatrix strips.

The second main novelty we present is that, in order to find a global weak solution, in general the rhs $f(t,x)$ should grow at most like $O(e^{\varepsilon|x|})$, for $|x| \to \infty$ uniformly in the separatrix strips S_j.

Finally, we derive sharp estimates on the singularities of the global solutions $u(t,x)$ of (5.1) near t_j, $j \in I_L$ for large classes of smooth rhs f, where

$$I_L = \{t_j : S_j \text{ or } S_{j-1} \text{ is separatrix}, j = 1, \dots, N\}.$$

We point out that the part ii) of Theorem 5.0.7 implies that L is not surjective in $C^\infty(\mathbb{R}^2)$ if and only if I_L is not empty.

In order to state the main result on the global solvability of (5.1) we introduce the subspace of the functions of infra-exponential growth in the x variable (e.g. see [51] where such growth plays an important role in theory of Fourier transform for hyperfunctions).

$$C^\infty(\mathbb{R} : Exp_{sl}(\mathbb{R})) \stackrel{\text{def}}{=}$$

$$\{f \in C^\infty(\mathbb{R}^2) : \forall T > 0, \forall \varepsilon > 0, \forall \alpha \in \mathbb{Z}^2_+, \exists C > 0 \ s.t. \ |\partial^\alpha_{t,x} f(t,x)| \le Ce^{\varepsilon|x|}, \ |t| \le T, x \in \mathbb{R}\}$$

We recall also the weighted Sobolev spaces $H^{s_1,s_2}(\mathbb{R}^n)$ in $\mathbb{R}^n$ (e.g. see [52]).

$$H^{s_1,s_2}(\mathbb{R}^n) \stackrel{\text{def}}{=} \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{s_1,s_2} = \|\langle x\rangle^{s_2}\langle D\rangle^{s_1} f\|_{L^2} < +\infty\}$$

which measure the global regularity and the behaviour on ∞ in $\mathbb{R}^n$, where $\langle x\rangle = \sqrt{1 + \|x\|^2}$.

Theorem 5.0.8. *Let L defined above be non-surjective in $C^\infty(\mathbb{R}^2)$. Then we can find a right inverse L^{-1} of L acting continuously*

$$L^{-1} : C^\infty(\mathbb{R} : Exp_{sl}(\mathbb{R})) \longrightarrow L^1_{loc}(\mathbb{R} : Exp_{sl}(\mathbb{R})) \bigcap C^\infty(\mathbb{R} \setminus I_L : Exp_{sl}(\mathbb{R}))$$

and

$$L^{-1} : C(\mathbb{R} : H^{s_1,s_2}(\mathbb{R})) \longrightarrow L^1_{loc}(\mathbb{R} : H^{s_1,s_2}(\mathbb{R})) \bigcap C(\mathbb{R} \setminus I_L : H^{s_1,s_2}(\mathbb{R})), \quad (5.5)$$

with $s_1, s_2 \in \mathbb{R}$.

Moreover, for any $\varepsilon > 0$ we have

$$\sup_{t\in[-\theta,\theta]} \left(\prod_{j=1}^{N} |t - t_j|^\varepsilon \|L_j^{-1} f(t,\cdot)\|_{H^{s_1,s_2}(\mathbb{R})} \right) \le C_{\varepsilon,s_1,s_2,\theta} \|f\|_{C(\overline{I}_j : H^{s_1,s_2}(\mathbb{R}))}$$

Next, if f is a polynomial function with respect to x, i.e.,

$$f(t,x) = \sum_{\ell=0}^{k} f_\ell(t) x^\ell ,$$

then

$$L^{-1} f(t,x) = \sum_{\ell=0}^{k} g_\ell(t) x^\ell$$

with

$$\begin{array}{lll} g_k(t) & = O(\ln^{k+1}|t-t_j|) & \textit{near } t=t_j,\ \textit{if } S_j \textit{ or } S_{j-1} \textit{ is a separatrix} \\ g_\ell(t) & = o(\ln^{k+1}|t-t_j|) & \textit{near } t=t_j,\ \textit{if } S_j \textit{ or } S_{j-1} \textit{ is a separatrix,} \end{array}$$

for $\ell = 0, \dots, k-1$.

Finally, given a zero order PDO $b(t,x,D)$ *in* x *smoothly depending on* t, *and* $s_1, s_2 \in \mathbb{R}$ *we can find* $\varepsilon_0 = \varepsilon_0(L, s_1, s_2) > 0$ *such that if*

$$\max_{\substack{|\alpha|\le[s_1]+2\\|\beta|\le[s_2]+2}} \sup_{\substack{t\in[t_j,t_{j+1}]\\(x,\xi)\in\mathbb{R}^2}} \langle x\rangle^{-\alpha}\langle\xi\rangle^{-\beta}|\partial_x^\alpha\partial_\xi^\beta b(t,x,\xi)| < \varepsilon_0$$

then $L + b(t,x,D)$ *admits a right inverse which satisfies* (5.5).

The results of this chapter will be published, jointly with T. Gramchev and A. Kirilov, in [5].

5.1 Separatrix Strips and Non-surjectivity

In this section we prove Theorem 5.0.7. We start by calculating the global "singular" characteristics of L after dividing by $p(t)$, namely, rewriting formally $Lu + bu = f$ to

$$\tilde{L}u + \frac{1}{p(t)} b(t,x,D)u = \frac{f(t,x)}{p(t)}$$

with

$$\tilde{L}u = \partial_t u + \frac{q(t)}{p(t)}\partial_x u$$

The characteristics of $\tilde{L}$, different from $t = t_j$, $j = 1, \dots, N$, are defined by

$$\dot{x}(t) = \frac{q(t)}{p(t)}, \qquad x|_{t=\tau} = y$$

for some $\tau \ne t_j$, $j = 1, \dots, N$.

We have

Lemma 5.1.1. *The function* $q(t)/p(t)$ *has a global primitive* $g(t)$ *such that*

$$g(t) = \sum_{j=1}^{N} \varkappa_j q(t_j) \ln|t-t_j| + \tilde{g}(t) \tag{5.6}$$

where each $\varkappa_j \in \mathbb{R} \setminus \{0\}$*, with* $j = 1, \ldots, N$*, depends only on* $p(t)$ *and* $\tilde{g} \in C^\infty(\mathbb{R})$*.*
Moreover, for each $j \in \{1, \ldots, N-1\}$ *fixed, we have*

$$\begin{aligned} \varkappa_j \varkappa_{j+1} q(t_j) q(t_{j+1}) > 0 \quad &\Leftrightarrow \quad q \text{ admits a zero in }]t_j, t_{j+1}[\text{ of odd order} \\ \varkappa_j \varkappa_{j+1} q(t_j) q(t_{j+1}) < 0 \quad &\Leftrightarrow \quad q \text{ does not admit zero of odd order} \end{aligned}$$

Proof. By the hypotheses (5.2), (5.3) on p and the decomposition of rational functions, there are nonzero real numbers $\varkappa_1, \ldots, \varkappa_N$ and $r_1 \in C^\infty(\mathbb{R})$ such that

$$\frac{1}{p(t)} = \sum_{j=1}^{N} \frac{\varkappa_j}{t - t_j} + r_1(t)$$

which yields

$$\frac{q(t)}{p(t)} = \sum_{j=1}^{N} \frac{\varkappa_j q(t_j)}{t - t_j} + r_2(t)$$

for some $r_2 \in C^\infty(\mathbb{R})$. The expression (5.6) follows by integration.

We note that the hypothesis (5.2) implies $q(t_j) \neq 0$, and hence

$$c_j \stackrel{\text{def}}{=} \varkappa_j q(t_j) \neq 0, \qquad j = 1, \ldots, N$$

□

Next, we present an important auxiliary result.

Lemma 5.1.2. *Let* $x(t, y)$ *be defined by*

$$\dot{x} = \frac{\lambda (t - \theta)^k}{(\theta_+ - t)(t - \theta_-)} + \tilde{q}(t), \quad x(\theta) = y, \quad \theta \in]\theta_-, \theta_+[, \tag{5.7}$$

with $\tilde{q} \in C^\infty([\theta_-, \theta_+])$*.*

Then one can find $r \in C^\infty([\theta_-, \theta_+])$ *such that*

$$x(t, y) = y + c_+ \ln |t - \theta_+| + c_- \ln |t - \theta_-| + r(t),$$

where

$$c_\pm = \mp \frac{\lambda (\theta_\pm - \theta)^k}{\theta_+ - \theta_-}$$

In particular, we observe that

i) $c_+ c_- > 0 \;\Leftrightarrow\; k$ *is odd* $\Leftrightarrow\; c_+$ *and* c_- *have the same signal and* $\lambda > 0$*;*

ii) $c_+ c_- < 0 \;\Leftrightarrow\; k$ *is even* $\Leftrightarrow\; c_+$ *and* c_- *have different signals and* $\lambda < 0$*.*

Proof. The proof follows from the decomposition

$$\frac{\lambda(t-\theta)^k}{(\theta_+ - t)(t-\theta_-)} = \frac{\lambda(\theta_+ - \theta)^k}{(\theta_+ - \theta_-)(\theta_+ - t)} + \frac{\lambda(\theta_\pm - \theta)^k}{(\theta_+ - \theta_-)(t-\theta_+)} + \tilde{q}_1(t),$$

where $\tilde{q}_1 = 0$ if $k = 0, 1$, and $\tilde{q}_1$ is polynomial of degree $k-2$, if $k \geq 2$, and integration (from θ to t) of the rhs of (5.7). □

Now we present the main steps of the proof of Theorem 5.0.7. First, assume that S_j is a separatrix, for some $j \in \{1, \ldots, N-1\}$. In view of Lemmas 5.1.1 and 5.1.2, the characteristic curves of L, in S_j, can be written in the form:

$$x(t,y) = y + c_j \ln|t - t_j| + c_{j+1} \ln|t - t_{j+1}| + R_j(t). \tag{5.8}$$

with $R_j \in C^\infty([t_j, t_{j+1}])$ and $c_j c_{j+1} > 0$. We observe that $c_j c_{j+1} > 0$ leads to

$$\lim_{t \to t_j^+} x(t,y) = \lim_{t \to t_{j+1}^-} x(t,y) = \mathrm{sign}(c_j)\infty, \qquad y \in \mathbb{R}. \tag{5.9}$$

Clearly (5.9) implies that every smooth curve with endpoints on $t = t_j$ and $t = t_{j+1}$ is hit at least twice by the characteristic curve (5.8) provided $y \gg 1$ (respectively, $-y \gg 1$) if $c_j > 0$ (respectively, $c_j < 0$), and therefore, the condition of Duistermaat-Hörmander for the surjectivity fails.

Suppose now that there are no separatrix strips. Hence, $p(t)$ and $q(t)$ do not change sign in $[t_j, t_{j+1}]$, $j = 0, 1, \ldots, N$, $t_0 = -\infty$, $t_{N+1} \overset{\text{def}}{=} +\infty$ and fixing j, we note that the line segment $x + \nu t = C$, $t \in [t_j, t_{j+1}]$ is transversal to L provided $\nu \neq 0$ has the same sign as $p(t)q(t)$ for some $t \in]t_j, t_{j+1}[$. So we have global piecewise smooth global transversal. Smoothing by mollifiers $\varepsilon^{-1}\varphi(\varepsilon^{-1}t)$ near $t = t_j$ makes the curve smooth and still globally transversal provided $0 < \varepsilon \ll 1$. The proof of Theorem 1.1 is complete.

Example 5.1.3. We focus on the vector fields $L_{\lambda,k}$ defined in (5.4) and exhibit some geometric features. The integral trajectories of $L_{\lambda,k}$ are given by the curves

$$x(t) = \lambda \left[(-1)^k \frac{1}{2} \log|1+t| - \frac{1}{2} \log|1-t| - \overline{\sum_{i<k}} \frac{t^i}{i} \right]$$

where $\overline{\sum}$ extends only to odd numbers when k is even and only to even numbers when k is odd.

The vector fields $L_{\lambda,k}$ are intrinsically Hamiltonian vector fields, i.e. they are tangent to the level sets of a regular smooth function on the plane – equivalently, the kernel of each operator $L_{\lambda,k}$ contains regular smooth functions.

For example, the following smooth function $f_{\lambda,k} \in \ker(L_{\lambda,k})$:

$$\begin{aligned} f_{\lambda,2k+1}(x,t) &= (1-t^2)\exp\left[2\left(\frac{x}{\lambda} + \overline{\sum_{i<2k+1}} \frac{t^i}{i}\right)\right], \text{ and} \\ f_{\lambda,2k}(x,t) &= \tan^{-1}\left\{\frac{1-t}{1+t}\exp\left[2\left(\frac{x}{\lambda} + \overline{\sum_{i<2k}} \frac{t^i}{i}\right)\right]\right\}. \end{aligned}$$

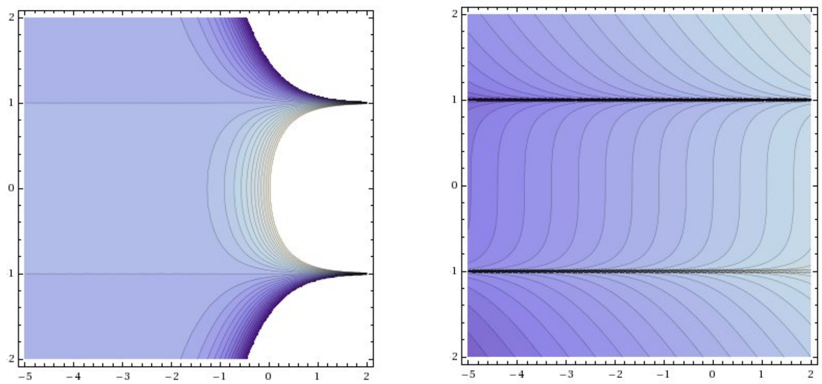

Figure 5.1: Integral curves of $L_{1,1} = (1-t^2)\partial_t + t\partial_x$ and $L_{1,2} = (1-t^2)\partial_t + t^2\partial_x$, respectively. Clearly no global transversal exists for $L_{1,1}$ while $L_{1,2}$ is topologically equivalent to a constant vector field.

Remark 5.1.4. We can generalize Theorem 1.2 for smooth non-singular vector fields assuming that p and q are in general position with respect to each other, i.e., each zero of p and q has finite multiplicity. Choosing t_1 and t_2 to be two successive zeros of $p(t)$, then t_1 and t_2 form a separatrix if and only if the sum of degrees of all the roots of q between t_1 and t_2 is odd.

5.2 Estimates on the right inverse

The aim of this section is to prove the Theorem 5.0.8. First we will construct a right inverse as follows:

Let $j \in \{1, \dots, N-1\}$. If the strip S_j is a separatrix, we use Lemma 5.1.1 to obtain

$$\begin{aligned} L_j^{-1} f &= \int_{\theta_j}^{t} \frac{f(\tau, x + g(\tau) - g(t))}{p(\tau)}\, d\tau \\ &= \int_{\theta_j}^{t} \frac{f(\tau, c_j \ln\frac{|\tau - t_j|}{|t-t_j|} + c_{j+1}\ln\frac{|\tau - t_{j+1}|}{|t - t_{j+1}|} + R_j(\tau) - R_j(t))}{p(\tau)}\, d\tau \end{aligned}$$

If S_j is not separatrix, we construct L_j^{-1} as the Green function for the Cauchy problem in S_j

$$L_j^{-1} f(t,x) = G_j^{\nu} f(t,x),$$

where $\nu \neq 0$ is fixed by the requirement $C_j : x + \nu t = 0$, $t \in [t_j, t_{j+1}]$ is non-characteristic for L in S_j and $u_j(t,x) = G_j^{\nu} f(t,x)$ is defined by

$$Lu_j = f, \quad (t,x) \in S_j, \quad u|_{C_j} = 0.$$

The global transversality of C_j in S_j implies that $u_j \in C^{\infty}(\overline{S}_j)$ (we are in a particular case of [20]).

The next assertion plays a crucial role in the proof of the global solvability for L in the presence of the separatrix strip.

Proposition 5.2.1. *Suppose that S_j is a separatrix and set $I_j \overset{\text{def}}{=} (t_j, t_{j+1})$, then L_j^{-1} has the following properties:*

i) If $C^{\infty}(I_j : E^{\varepsilon}_{gr}(\mathbb{R}))$ (respectively, $C^{\infty}(I_j : E^{\varepsilon}_{dec}(\mathbb{R}))$) is the subspace of $C^{\infty}(I_j \times \mathbb{R})$ consisting of all infinitely differentiable functions that satisfy the following growth (respectively, decay) condition

$$\forall \alpha \in \mathbb{Z}^2_+, \exists C > 0 \text{ such that } |\partial^{\alpha}_{t,x} f(t,x)| \leq Ce^{\varepsilon |x|},\ t \in I_j, x \in \mathbb{R}$$

(respectively,

$$\forall \alpha \in \mathbb{Z}^2_+, \exists C > 0 \text{ such that } |\partial^{\alpha}_{t,x} f(t,x)| \leq Ce^{-\varepsilon |x|},\ t \in I_j, x \in \mathbb{R})$$

then

$$L_j^{-1} : C^{\infty}(\overline{I}_j : E^{\varepsilon}_{gr}(\mathbb{R})) \longrightarrow L^1(I_j : E^{\varepsilon}_{gr}(\mathbb{R})) \bigcap C^{\infty}(I_j : E^{\varepsilon}_{gr}(\mathbb{R})) \tag{5.10}$$

(respectively,

$$L_j^{-1} : C^{\infty}(\overline{I}_j : E^{\varepsilon}_{dec}(\mathbb{R})) \longrightarrow L^1(I_j : E^{\varepsilon}_{dec}(\mathbb{R})) \bigcap C^{\infty}(I_j : E^{\varepsilon}_{dec}(\mathbb{R})))$$

if

$$0 < \varepsilon < \min\{|c_j|^{-1}, |c_{j+1}|^{-1}\}$$

ii) For $s_1, s_2 \in \mathbb{R}$,

$$L_j^{-1} : C(\overline{I}_j : H^{s_1,s_2}(\mathbb{R})) \longrightarrow L^1(I_j : H^{s_1,s_2}(\mathbb{R})) \bigcap C(I_j : H^{s_1,s_2}(\mathbb{R})) \tag{5.11}$$

Moreover, for any $\varepsilon > 0$ we have

$$\sup_{t \in [t_j, t_{j+1}]} \left(|t - t_j|^{\varepsilon} |t - t_{j+1}|^{\varepsilon} \| L_j^{-1} f(t,\cdot) \|_{H^{s_1,s_2}(\mathbb{R})} \right) \leq C_{\varepsilon, s_1, s_2} \|f\|_{C(\overline{I}_j : H^{s_1,s_2}(\mathbb{R}))} \tag{5.12}$$

iii) If $f(t,x) = \sum_{\ell=0}^{k} f_{\ell}(t) x^{\ell}$, then

$$L_j^{-1} f(t,x) = \sum_{\ell=0}^{k} g_{\ell}(t) x^{\ell}$$

with

$$\begin{cases} g_k(t) & = g_k(t_{\mu}) \gamma_{\mu} \ln^{k+1} \frac{1}{|t - t_{\mu}|} (1 + o(1)) \text{ near } t = t_{\mu},\ \gamma_{\mu} \neq 0,\ \mu = j, j+1, \\ g_{\ell}(t) & = o(\ln^{k+1} \frac{1}{|t - t_{\mu}|}) \text{ near } t = t_{\mu},\ \mu = j, j+1, \quad \ell = 0, 1, \ldots, k-1. \end{cases}$$

iv) Given a zero order PDO $b(t,x,D)$ in x, smoothly depending on t, and $s_1, s_2 \in \mathbb{R}$, we can find $\varepsilon_0 = \varepsilon_0(L, s_1, s_2) > 0$ such that if

$$\max_{\substack{|\alpha|\le[s_1]+2\\|\beta|\le[s_2]+2}} \sup_{\substack{t\in[t_j,t_{j+1}]\\(x,\xi)\in\mathbb{R}^2}} \langle x\rangle^{-\alpha}\langle\xi\rangle^{-\beta}|\partial_x^\alpha\partial_\xi^\beta b(t,x,\xi)| < \varepsilon_0$$

then $L + b(t,x,D)$ admits a right inverse which satisfies (5.11)*.*

Proof. We observe that for t close to t_j we can write

$$L_j^{-1}f(t,x) \quad = \quad \int_{\theta_j}^{t} f(\tau, c_j \ln\frac{|\tau-t_j|}{|t-t_j|} + M_j(t,\tau))\frac{f_j(\tau)}{\tau-t_j}\,d\tau \tag{5.13}$$

with $f_j \in C^\infty([t_j,\theta_j])$, $M_j \in C^\infty(\Delta_j)$, $\Delta_j = \{t_j \le \tau \le t \le \theta_j\}$. Therefore,

$$\begin{aligned}
|\partial_x^\alpha L_j^{-1}f(t,x)| &\le \int_t^{\theta_j} |\partial_x^\alpha f(\tau, x + c_j\ln\frac{|\tau-t_j|}{|t-t_j|} + M_j(t,\tau))\frac{f_j(\tau)}{\tau-t_j}|\,d\tau\\
&\le Ce^{\varepsilon|x|}\int_t^{\theta_j} e^{\varepsilon|c_j|\ln\frac{\tau-t_j}{t-t_j}}\frac{1}{\tau-t_j}\,d\tau\\
&= Ce^{\varepsilon|x|}\frac{1}{(t-t_j)^{\varepsilon|c_j|}}\int_t^{\theta_j}\frac{1}{(\tau-t_j)^{1-\varepsilon|c_j|}}\,d\tau\\
&= Ce^{\varepsilon|x|}\frac{1}{\varepsilon|c_j|(t-t_j)^{\varepsilon|c_j|}}((\theta_j-t_j)^{\varepsilon|c_j|} - (t-t_j)^{\varepsilon|c_j|})\\
&= Ce^{\varepsilon|x|}\frac{(\theta_j-t_j)^{\varepsilon|c_j|}}{\varepsilon|c_j|(t-t_j)^{\varepsilon|c_j|}}(1 + O((t-t_j)^{\varepsilon|c_j|}))
\end{aligned} \tag{5.14}$$

Similarly, we derive that near t_{j+1} we have

$$|\partial_x^\alpha L_j^{-1}f(t,x)| \le Ce^{\varepsilon|x|}\frac{(t_{j+1}-\theta_j)^{\varepsilon|c_{j+1}|}}{\varepsilon|c_{j+1}|(t_{j+1}-t)^{\varepsilon|c_{j+1}|}}(1 + O((t_{j+1}-t)^{\varepsilon|c_{j+1}|})) \tag{5.15}$$

Clearly, (5.13), (5.14), (5.15) imply (5.10) provided $0 < \varepsilon < \min\{\frac{1}{|c_j|}, \frac{1}{|c_{j+1}|}\}$.
As it concerns to item *ii)*, taking into account the inequality

$$\sup_{x\in\mathbb{R},|\lambda|\ge1} |\lambda|^{-|s_2|}\langle x\rangle^{s_2}\langle x+\lambda\rangle^{-s_2} \quad < \quad +\infty$$

we observe that for $\alpha \in \mathbb{Z}_+$ and $s_2 \in \mathbb{R}$ we have for t near t_j

$$\begin{aligned}
\|\langle\cdot\rangle^{s_2}\partial_x^\alpha L_j^{-1} f(t,\cdot)\|_{L^2} &\leq C \int_t^{\theta_j} \sup_{x\in R}\left(\langle x\rangle^{s_2}\langle x + c_j \ln\tfrac{|\tau-t_j|}{|t-t_j|}\rangle^{-s_2}\right)\tfrac{1}{|\tau-t_j|}\,d\tau \\
&\quad\times \sup_{t\in[t_j,t_{j+1}]}\|\langle\cdot\rangle^{s_2}\partial^\alpha f(t,\cdot)\|_{L^2} \\
&\leq \tilde{C}\int_{\theta_j}^{t} \ln^{|s_2|}\left(\tfrac{\tau-t_j}{t-t_j}\right)\tfrac{1}{\tau-t_j}\,d\tau \sup_{t\in[t_j,t_{j+1}]}\|\langle\cdot\rangle^{s_2}\partial^\alpha f(t,\cdot)\|_{L^2} \\
&= \frac{\tilde{C}}{|s_2|}\ln^{|s_2|+1}\frac{1}{t-t_j}\sup_{t\in[t_j,t_{j+1}]}\|\langle\cdot\rangle^{s_2}\partial^\alpha f(t,\cdot)\|_{L^2} \quad (5.16)
\end{aligned}$$

Therefore we obtained (5.11) for $s_1 \in \mathbb{Z}_+$ (summation in (5.16) over $|\alpha|$). We conclude the general case for s_1 by interpolation and duality arguments.

Since the logarithmic singularity is weaker then any polynomial one, (5.16) yields (5.12). □

Next, we show a gluing lemma, which will imply that

$$L^{-1}f(t,x) \quad = \quad L_j^{-1}f(t,x), \qquad (t,x) \in S_j,\ j = 0,1,\dots,N$$

is a right inverse satisfying the properties stated in Theorem 1.2. This gluing auxiliary assertion seems to be also a novelty "per se" and might be of an independent interest.

Let Ω be an open domain in $\mathbb{R}^n$ and let $\delta > 0$. Set $I_\delta = (-\delta,\delta)$, $I_\delta^+ =]0,\delta[$, $I_\delta^- =]-\delta,0[$, and

$$\begin{aligned}
\Omega_\delta^\pm = I_\delta^\pm \times \Omega = \{(t,x) :\ 0 < \pm t < \delta, x \in \Omega\}, \\
\Omega_\delta = I_\delta \times \Omega = \{(t,x) :\ |t| < \delta, x \in \Omega\}.
\end{aligned}$$

Consider the smooth vector field

$$X = a(t,x)\partial_t + \sum_{j=1}^n a_j(t,x)\partial_{x_j},$$

having $t = 0$ as a characteristic, i.e.,

$$a_0(0,x) = 0, \quad x \in \Omega$$

Let

$$b = b(t,x) \in C^\infty(\Omega_\delta)$$

or, in the case $\Omega = \mathbb{R}^n$, we allow b to be a zero order PDO in x (see [52]) depending smoothly on $t \in]-\delta,\delta[$.

We have:

Lemma 5.2.2. *Let $f \in C^\infty(\Omega_\delta)$ (respectively, $f \in C(\,(-\delta,\delta) : H^{s_1,s_2}(\mathbb{R}^n))$ for some $s_1, s_2 \in \mathbb{R}$ if $\Omega = \mathbb{R}^n$). Suppose that*

$$u^\pm \quad \in \quad C^\infty(\Omega_\delta^\pm)$$

(respectively,

$$u^\pm \quad \in \quad C(I_\delta^\pm : H^{s_1,s_2}(\mathbb{R}^n))$$

for some $s_1, s_2 \in \mathbb{R}$) satisfies

$$Xu^\pm + bu^\pm = f \quad \text{in } \Omega_\delta^\pm$$

Then

$$u(t,x) = \begin{cases} u^+(t,x) & \text{if} \quad (t,x) \in \Omega_\delta^+ \\ u^-(t,x) & \text{if} \quad (t,x) \in \Omega_\delta^- \end{cases}$$

is a well defined $L^1_{loc}(\Omega)$ (respectively, $L^1(I_\delta : H^{s_1,s_2}(\mathbb{R}^n))$ distributional solution of $Xu = f$ in Ω_δ provided

$$u^\pm \quad \in \quad L^1(I_\delta^\pm \times K), \qquad K \subset\subset \Omega \tag{5.17}$$

(respectively,

$$u^\pm \quad \in \quad L^1(I_\delta^\pm : H^{s_1,s_2}(\mathbb{R}^n)), \tag{5.18}$$

if $\Omega = \mathbb{R}^n$)) and

$$\lim_{t\to 0^\pm} \int_{\mathbb{R}^n} a(t,x)u^\pm(t,x)\varphi(t,x)dx = 0, \qquad \varphi \in C_0^\infty(\Omega_\delta) \tag{5.19}$$

Proof. Let $\varphi(t,x) \in C_0^\infty(\Omega_\delta)$. We have to prove that

$$\langle u, X^*\varphi + b^*\varphi\rangle = \langle f, \varphi\rangle \tag{5.20}$$

where X^* (respectively, b^*) stands for the adjoint of X (respectively, b).

Taking into account (5.17), (5.18) and Lebesgue's dominated convergence theorem we have

$$\langle u, X^*\varphi + b^*\varphi\rangle = \lim_{\varepsilon\to 0}(J_\varepsilon^+(u^+,\varphi) + J_\varepsilon^-(u^-,\varphi)),$$

where

$$J_\varepsilon^\pm(u^\pm,\varphi) = \pm\int_{\pm\varepsilon}^{\pm\delta}\left(\int_\Omega u^\pm(t,x)(X^*\varphi(t,x) + b^*(t,x,D)\varphi(t,x))dx\right)dt.$$

Integration by parts, duality arguments, the Fubini theorem and (5.17) imply that

$$\begin{aligned} J_\varepsilon^\pm(u^\pm,\varphi) &= \pm\int_{\pm\varepsilon}^{\pm\delta}\int_\Omega (Xu^\pm(t,x)+b(t,x,D)u^\pm(t,x))\varphi(t,x)dxdt \\ &\quad + \int_\Omega a(\pm\varepsilon,x)u^\pm(\pm\varepsilon,x)\varphi(\pm\varepsilon,x)dx \\ &= \int_{\Omega_\delta^\pm\setminus\overline{\Omega_\varepsilon^\pm}} f(t,x)\varphi(t,x) \\ &\quad + \int_\Omega a(\pm\varepsilon,x)u^\pm(\pm\varepsilon,x)\varphi(\pm\varepsilon,x)dx \end{aligned}$$

Next, using the hypothesis (5.19), we deduce that

$$\lim_{\varepsilon\to 0} J_\varepsilon^\pm(u^\pm,\varphi) = \int_{\Omega_\delta^\pm} f(t,x)\varphi(t,x)dtdx$$

and, plugging into the rhs of (5.20), we obtain,

$$\begin{aligned} \langle u, L^*\varphi+b^*\varphi\rangle &= \int_{\Omega_\delta^+} f(t,x)\varphi(t,x)dtdx + \int_{\Omega_\delta^-} f(t,x)\varphi(t,x)dtdx \\ &= \int_{\Omega_\delta} f(t,x)\varphi(t,x)dtdx \end{aligned}$$

This completes the proof of the lemma. □

Combining Proposition 3.1 and Lemma 3.2 we derive the assertions for L^{-1}. As it concerns the perturbation with $b(t,x,D)$, we reduce the equation in $\mathbb{R}^2$ to $Lu+b(t,x,D_x)u=f$ on S_j, $j=0,1,\ldots,N$. We are reduced to the study of the global solvability of

$$u+L^{-1}b(t,x,D)u = L^{-1}f, \qquad (t,x)\in S_j, j=0,1,\ldots,N.$$

We apply the Picard type scheme

$$u_k = -L^{-1}b(t,x,D)u_{k-1}+L^{-1}f, \qquad k\in\mathbb{N}, u_0=0 \tag{5.21}$$

If $j=1,\ldots,N$, we use the results for H^{s_1,s_2} estimates of PDO in $\mathbb{R}^n$ (see [52]) and choose ε_0 so small that

$$\|b(t,x,D)L^{-1}\|_{L^1([t_1,t_N]:H^{s_1,s_2})\to L^1([t_1,t_N]:H^{s_1,s_2})} < 1$$

Using continuity arguments we can find $\delta > 0$ (small enough) such that

$$\|L^{-1}b(t,x,D)L^{-1}\|_{L^1([t_-\delta,t_N+\delta]:H^{s_1,s_2})\to L^1([t_1-\delta,t_N+\delta]:H^{s_1,s_2})} < 1$$

Since $p(t)$ has no zeroes for $t > t_N + \delta$ and $t \leq t_1 - \delta$ we have the following estimates: there exist a $C = C_\delta > 0$ such that

$$\|L^{-1}bu(t,\cdot)\|_{H^{s_1,s_2}} \leq C_\delta \int_{\theta_j}^{t} \|u(\tau,\cdot)\|_{H^{s_1,s_2}} d\tau,$$

for $j = 0$, $t \leq t_1 - \delta$, $j = N$, $t \geq t_N + \delta$. Combination of contraction and Gronwall inequalities (see [49]) imply the convergence of (5.21) and the existence of $(L+b)^{-1}$ satisfying the last part of Theorem 1.2.

Remark 5.2.3. *We point out that the estimates for $f \in C^\infty(\overline{I}_j : E^\varepsilon_{dec}(\mathbb{R}))$ allows to extend solvability for L and $L+b$ in Gelfand-Shilov spaces $S^\mu_\mu(\mathbb{R})$ in x, provided $\mu > 1$ (see [53] for global solvability and regularity results for some degenerate PDO under similar sub-exponential decay conditions). We can show that, if the decay is super-exponential the solution u loses this decay, unlike the solvability in Gelfand-Shilov spaces $S^\mu_\mu, 1/2 \leq \mu \leq 1$, (see [54, 55] and the references therein).*

5.3 The sharpness of the estimates for L_0

We consider the model equation $L_0 u = f$. Using the method of the characteristics, for $t \neq \pm 1$, one can write formally a right inverse of L_0 in the following way

$$L_0^{-1} f \stackrel{\text{def}}{=} \int_0^t f(\tau, x + \ln\left|\frac{1-\tau^2}{1-t^2}\right|)\frac{1}{1-\tau^2}\, d\tau = G_+ f + G_- f, \tag{5.22}$$

where

$$G_\pm f(t,x) \stackrel{\text{def}}{=} \frac{1}{2}\int_0^t f(\tau, x + \ln\left|\frac{1-\tau}{1-t}\right| + \ln\left|\frac{1+\tau}{1+t}\right|)\frac{1}{1\pm\tau}\, d\tau \tag{5.23}$$

We define in a natural way $C^\infty(\mathbb{R} : E^\varepsilon_{gr}(\mathbb{R}))$ as the inductive limit

$$C^\infty(\mathbb{R} : E^\varepsilon_{gr}(\mathbb{R})) = \lim_{T\nearrow+\infty} C^\infty(\,[T,T] : E^\varepsilon_{gr}(\mathbb{R}))$$

Observe that $C^\infty(\mathbb{R} : E^\varepsilon_{gr}(\mathbb{R}))$ is a vector subspace of $C^\infty(\mathbb{R}^2)$ and, given $f_1, f_2 \in C^\infty(\mathbb{R} : E^\varepsilon_{gr}(\mathbb{R}))$, we have $f_1 \cdot f_2 \in C^\infty(\mathbb{R} : E^\varepsilon_{gr}(\mathbb{R}))$. In particular, the projections $\pi_1(t,x) = t$ and $\pi_2(t,x) = x$ belong to this space and consequently, any polynomial function p belongs to $C^\infty(\mathbb{R} : E^\varepsilon_{gr}(\mathbb{R}))$.

We introduce a topology on $C^\infty(\mathbb{R} : E^\varepsilon_{gr}(\mathbb{R}))$ by the following family of seminorms

$$g^\varepsilon_{j,k,T}(f) \stackrel{\text{def}}{=} \sup\{\ |e^{-\varepsilon|x|}\partial_t^{\alpha_1}\partial_x^{\alpha_2} f(t,x)|; |\alpha_1| \leq j, |\alpha_2| \leq k, |t| \leq T, x \in \mathbb{R}, \}$$

where $T > 0$ and $j, k \in \mathbb{Z}_+$.

Lemma 5.3.1. *If $a \in C^1(\mathbb{R})$ and $p \in \mathbb{N}$ then, when $t \to 1$, we have*

$$\int_0^t a(s)\ln^p\left|\frac{1-s}{1-t}\right|\frac{1}{1-s}\,ds = \frac{a(1)}{p+1}\ln^{p+1}\left|\frac{1-s}{1-t}\right|(1+o(1))$$

Proof.

$$\begin{aligned}
\int_0^t a(s)\ln^p\left|\frac{1-s}{1-t}\right|\frac{1}{1-s}\,ds &= a(1)\int_0^t \ln^p\left|\tfrac{1-s}{1-t}\right|\tfrac{1}{1-s}\,ds + \int_0^t a_1(s)\ln^p\left|\tfrac{1-s}{1-t}\right|ds \\
&= \frac{a(1)}{p+1}\ln^{p+1}\left|\tfrac{1-s}{1-t}\right| + o\left(\ln^p\tfrac{1}{|1-t|}\right) \\
&= \frac{a(1)}{p+1}\ln^{p+1}\left|\tfrac{1}{1-t}\right|(1+o(1))
\end{aligned}$$

□

Lemma 5.3.2. *If f is a monomial function with respect to x, i.e., $f(t,x) = f_j(t)x^j$, with $f_j \in C^1(\mathbb{R})$ and $j \in \mathbb{Z}_+$, then*

$$L_0^{-1}f(t,x) = \sum_{\ell=0}^{j} g_{j\ell}(t)x^\ell$$

with

$$\begin{cases} g_{j0}(t) &= \frac{f_0(\pm1)}{2}\ln\left|\frac{1}{1\mp t}\right|(1+o(1)), t \to \pm1 \\ g_{j\ell}(t) &= O(\ \ln^{j+1-\ell}\frac{1}{|1\mp t|}\ ), t \to \pm1. \end{cases}$$

Proof. From (5.22) and (5.23) we obtain

$$\begin{aligned}
G_\pm f(t,x) &= \frac{1}{2}\int_0^t f_j(\tau)\left(x + \ln\left|\frac{1-\tau}{1-t}\right| + \ln\left|\frac{1+\tau}{1+t}\right|\right)^j\frac{1}{1\pm\tau}\,d\tau \\
&= \sum_{\ell=0}^{j}\left[\frac{1}{2}\binom{j}{\ell}\int_0^t f_j(\tau)\left(\ln\left|\frac{1-\tau}{1-t}\right| + \ln\left|\frac{1+\tau}{1+t}\right|\right)^{j-\ell}\frac{1}{1\pm\tau}\,d\tau\right]x^\ell \\
&= \sum_{\ell=0}^{j} g_{j\ell\pm}(t)\,x^\ell
\end{aligned}$$

where

$$\begin{aligned} g_{j\ell\pm}(t) &= \frac{1}{2}\binom{j}{\ell}\int\limits_0^t f_j(\tau)\left(\ln\left|\frac{1-\tau}{1-t}\right| + \ln\left|\frac{1+\tau}{1+t}\right|\right)^{j-\ell}\frac{1}{1\pm\tau}\,d\tau \\ &= \frac{1}{2}\binom{j}{\ell}\sum_{m=0}^{j-\ell}\binom{j-\ell}{m}\int\limits_0^t f_j(\tau)\ln^m\left|\frac{1-\tau}{1-t}\right|\ln^{j-\ell-m}\left|\frac{1+\tau}{1+t}\right|\frac{1}{1\pm\tau}\,d\tau \end{aligned}$$

Now, it follows from Lemma 5.3.1 that, near $t=1$, we have

$$\begin{aligned} g_{j\ell-}(t) &= \frac{1}{2}\binom{j}{\ell}\sum_{m=0}^{\ell}\binom{j-\ell}{m}\int\limits_0^t f_j(\tau)\ln^m\left|\frac{1-\tau}{1-t}\right|\ln^{j-\ell-m}\left|\frac{1+\tau}{1+t}\right|\frac{1}{1-\tau}\,d\tau \\ &= \frac{1}{2}\binom{j}{\ell}\sum_{m=0}^{j-\ell}\binom{j-\ell}{m}\frac{f_j(1)\ln^{j-\ell-m}\left|\frac{2}{1+t}\right|}{m+1}\ln^{m+1}\left|\frac{1}{1-t}\right|(1+o(1)) \\ &= O\left(\ln^{j+1-\ell}\left|\frac{1}{1-t}\right|\right) \end{aligned}$$

Analogously, near $t=-1$, we have

$$\begin{aligned} g_{j\ell+}(t) &= \frac{1}{2}\binom{j}{\ell}\sum_{m=0}^{j-\ell}\binom{j-\ell}{m}\int\limits_0^t f_j(\tau)\ln^m\left|\frac{1-\tau}{1-t}\right|\ln^{j-\ell-m}\left|\frac{1+\tau}{1+t}\right|\frac{1}{1+\tau}\,d\tau \\ &= \frac{1}{2}\binom{j}{\ell}\sum_{m=0}^{j-\ell}\binom{l}{m}\frac{f_j(1)\ln^m\left|\frac{2}{1-t}\right|}{j-\ell-m+1}\ln^{j-\ell-m+1}\left|\frac{1}{1+t}\right|(1+o(1)) \\ &= O\left(\ln^{j+1-\ell}\left|\frac{1}{1+t}\right|\right) \end{aligned}$$

In particular, for $\ell=0$, we have

$$g_{j0\pm}(t) = \frac{1}{2}\int\limits_0^t f_j(\tau)\frac{1}{1\pm\tau}\,d\tau = \frac{f_j(\pm 1)}{2}\ln\left|\frac{1}{1\mp t}\right|(1+o(1)).$$

□

The next assertion shows that we have sharp estimates on the singularities.

Proposition 5.3.3. *The following properties hold: there exists $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$*

i) Given $T>0$ and $k\in\mathbb{Z}_+$, we have

$$\sup_{\substack{x\in\mathbb{R}\\ |\alpha|\le k}} |(1-t^2)^{\varepsilon}L_0^{-1}f(t,\cdot)| \le C_T^{\varepsilon_0}\, g^{\varepsilon_0}_{0,k,T}(f)$$

ii) If $f(t,x)=\sum_{j=0}^{k} f_j(t)x^j$*, then*

$$L_0^{-1} f(t,x)=\sum_{j=0}^{k} g_j(t)x^j$$

with

$$\begin{cases} g_0(t) &= \frac{f_k(\pm 1)}{2(k+1)} \ln^{k+1}|1\pm t|(1+o(1)), t\to \pm 1, \\ g_j(t) &= O(\ln^{k+1-j}|1\pm t|), t\to \pm 1 \end{cases}$$

iii) $u=L_0^{-1}f$ *is a weak solution of* $Lu=f$ *for all* $f\in C^\infty(\mathbb{R}:E^\varepsilon(\mathbb{R}))$ *such that* $\forall\alpha\in\mathbb{Z}_+$, $K\subset\subset\mathbb{R}$*, there exist* $M>0$ *such that*

$$|\partial_x^\alpha u(t,x)|\le M|1\pm t|^{-\varepsilon}, \quad 0<|1\pm t|\ll 1,\, x\in K$$

Proof. To prove *i)* we start by defining, for each $T>0, k\in\mathbb{Z}_+$ and $u\in C^\infty(\mathbb{R}:E^\varepsilon(\mathbb{R}))$ the following function:

$$P_{k,T}^{\varepsilon_0}(u)\overset{\text{def}}{=}\int_{-T}^{T}\sup_{\substack{x\in\mathbb{R}\\ |\alpha|\le k}}\left|e^{-\varepsilon_0|x|}\partial_x^\alpha u(t,x)\right|dt$$

Thus, for any $f\in C^\infty(\mathbb{R}:E^\varepsilon(\mathbb{R}))$, with $0<\varepsilon<\varepsilon_0$ and $t>0$ we have

$$\begin{aligned} & P_{k,T}^{\varepsilon_0}(G_-f)=\int_{-T}^{T}\sup_{\substack{x\in\mathbb{R},\\ |\alpha|\le k}}\left|e^{-\varepsilon_0|x|}\partial_x^\alpha G_-f(t,x)\right|dt \\ = & \int_{-T}^{T}\sup_{\substack{x\in\mathbb{R},\\ |\alpha|\le k}}\left|e^{-\varepsilon_0|x|}\partial_x^\alpha\left[\frac{1}{2}\int_0^t f(\tau,x+\ln|\tfrac{1-\tau}{1-t}|+\ln|\tfrac{1+\tau}{1+t}|)\frac{1}{1-\tau}\,d\tau\right]\right|dt \\ \le & \frac{1}{2}\int_{-T}^{T}\int_0^t\sup_{\substack{x\in\mathbb{R},\\ |\alpha|\le k}}\left|e^{-\varepsilon_0|x|}\partial_x^\alpha f(\tau,x+\ln|\tfrac{1-\tau}{1-t}|+\ln|\tfrac{1+\tau}{1+t}|)\frac{1}{1-\tau}\,d\tau\right|dt \\ \le & \frac{1}{2}g_{0,k,T}^{\varepsilon}(f)\int_{-T}^{T}\int_0^t\sup_{\substack{x\in\mathbb{R},\\ |\alpha|\le k}}\left|e^{-\varepsilon_0|x|}\exp(\varepsilon(|x|+\ln|\tfrac{1-\tau}{1-t}|-\ln|\tfrac{1+\tau}{1+t}|\,))\frac{1}{1-\tau}\right|d\tau dt \\ \le & \frac{1}{2}\,e^{\varepsilon-\varepsilon_0}g_{0,k,T}^{\varepsilon}(f)\int_{-T}^{T}\int_0^t\left|\frac{1+\tau}{1+t}\right|^{-\varepsilon}\left|\frac{1-\tau}{1-t}\right|^{\varepsilon}\frac{1}{|1-\tau|}d\tau dt \\ \le & \frac{1}{2}\,C_T^{\varepsilon_0}\,g_{0,k,T}^{\varepsilon_0}(f)(1-t)^{-\varepsilon} \end{aligned}$$

By using the same arguments, when $t<0$, we obtain an analogous estimate to G_+f, and consequently

$$P_{k,T}^{\varepsilon_0}(L_0^{-1}f(t,x))\le C_T^{\varepsilon_0}\,g_{0,k,T}^{\varepsilon_0}(f)(1-t^2)^{-\varepsilon}$$

To prove *ii)*, we use the results in the lemmas 5.3.1 and 5.3.2 below to obtain

$$\begin{aligned} L_0^{-1} f(t,x) &= \sum_{\ell=0}^{k} L_0^{-1}(f_j(t)x^j) = \sum_{\ell=0}^{k} \left(\sum_{\ell=0}^{j} g_{j\ell}(t) x^\ell \right) \\ &= \sum_{\ell=0}^{k} \left(\sum_{\ell=j}^{k} g_{\ell j}(t) \right) x^\ell = \sum_{\ell=0}^{k} g_j(t) x^\ell \end{aligned}$$

where

$$g_j(t) \stackrel{\text{def}}{=} \sum_{\ell=j}^{k} g_{\ell j}(t) = \sum_{\ell=j}^{k} O\left(\ln^{\ell+1-j} \frac{1}{|1 \mp t|} \right) = O\left(\ln^{k+1-j} \frac{1}{|1 \mp t|} \right), \text{ when } t \to \pm 1$$

To prove the statement *iii)*, first, for $0 < t < 1$, we have

$$\begin{aligned} |\partial_x^\alpha (G_- f(t,x))| &\leq \frac{1}{2} \int_0^t \left| \partial_x^\alpha f(\tau, x + \ln|\tfrac{1-\tau}{1-t}| + \ln|\tfrac{1+\tau}{1+t}|) \frac{1}{1-\tau} \right| d\tau \\ &\leq \frac{1}{2}\, g^\varepsilon_{0,\alpha,T}(f)\, e^{\varepsilon|x|} \int_0^t |\frac{1-\tau}{1-t}|^\varepsilon |\frac{1+\tau}{1+t}|^{-\varepsilon} \frac{1}{|1-\tau|} d\tau \\ &\leq \frac{1}{2}\, g^\varepsilon_{0,\alpha,T}(f)\, e^{\varepsilon|x|} \varepsilon^{-1} |1-t|^{-\varepsilon} \end{aligned}$$

By using the same arguments, when $-1 < t < 0$, we obtain the same estimate to $G_+ f$. Therefore

$$|\partial_x^\alpha (G_+ f(t,x))| \leq \frac{1}{2}\, g^\varepsilon_{0,\alpha,T}(f)\, e^{\varepsilon|x|} \varepsilon^{-1} |1+t|^{-\varepsilon}$$

Therefore, given $f \in C^\infty(\mathbb{R} : E^\varepsilon(\mathbb{R})), \alpha \in \mathbb{Z}_+$ and $K \subset\subset \mathbb{R}$, we set

$$M = \varepsilon^{-1}\, g^\varepsilon_{0,\alpha,T}(f) \sup_{x \in K} e^{\varepsilon|x|}.$$

Thus, it follows from (5.22) and (5.23) that

$$|\partial_x^\alpha (L_0^{-1} f(t,x))| \leq M\, |1 \pm t|^{-\varepsilon}, \quad 0 < |1 \pm t| \ll 1,\, x \in K$$

□

Remark 5.3.4. *Since the general solution of $L_0 u = f$ in $[-1,1] \times \mathbb{R}$ is given by*

$$u = \varphi(x + \ln(1-t^2)) + L_0^{-1} f(t,x),$$

with φ being a function (or distribution) of one variable, we observe that we have always singularity at $t = -1$ or $t = +1$.

In view of the separatrix phenomena, we have not compensate both singularities in the general case, while we can "cancel" the singularity either at $t = -1$ or $t = +1$.

If $f \equiv c \neq 0$, we exhibit, apart from $u = \frac{c}{2}\ln|\frac{1+t}{1-t}|$, two particular solutions:

$$u_\pm(t,x) = \mp\frac{c}{2}x \pm c\ln|1 \mp t|$$

5.4 Perturbation with non-degenerate PDOs

The aim of this section is to show that if we perturb L_0 with constants PDOs, or more generally, a Fourier multiplier satisfying suitable non-degeneracy conditions, we can obtain L^∞_{loc} estimates in t for the $(L_0 + b)^{-1}$ without the smallness requirement on b.

More precisely, we consider

$$L_b u = (1-t^2)\partial_t u - 2t\partial_x u + b(D)u = f(t,x)$$

where

$$b(\xi) \in C(\mathbb{R}) \text{ is real-valued and bounded away from zero for } \xi \in \mathbb{R}. \tag{5.24}$$

Clearly (5.24) implies that one can find $0 < \delta_0 < \delta_1$ such that

$$\text{either } \delta_0 \leq b(\xi) \leq \delta_1 \;\text{ or }\; -\delta_1 \leq b(\xi) \leq -\delta_0, \text{ for } \xi \in \mathbb{R}. \tag{5.25}$$

Set $\hat{u}(t,\xi) = \mathcal{F}_{x\to\xi}u(t,\cdot)$ to be the partial Fourier transform in x, i.e.,

$$\hat{w}(\xi) = \int_{\mathbb{R}} e^{-ix\xi}w(x)dx.$$

Setting (formally)

$$\hat{u}(t,\xi) = \exp(-\frac{b(\xi)}{2}\ln\left|\frac{1-t}{1+t}\right|) = \left(\left|\frac{1-t}{1+t}\right|\right)^{b(\xi)/2}\hat{v}(t,\xi)$$

we obtain that

$$\hat{L_0}\hat{v}(t,\xi) = \left(\left|\frac{1-t}{1+t}\right|\right)^{-b(\xi)/2}\hat{f}(t,\xi).$$

In view of the non-degeneracy condition (5.25) can write a right inverse of L_b which is L^∞_{loc} in t (a better regularity than L^1_{loc} for L^{-1}). Indeed, set

$$\hat{L_b}^{-1}\hat{f} \;=\; \left(\left|\frac{1-t}{1+t}\right|\right)^{b(\xi)/2}\int_{-\mathrm{sign}(b)}^{t}\frac{\hat{f}(s,\xi)e^{i\ln\frac{|1-t^2|}{|1-\tau^2|}}}{(1-\tau)|1-\tau|^{b(\xi)/2}(1+s)|1+s|^{-b(\xi)/2}}\,ds$$

Proposition 5.4.1. *The operator L_b^{-1} acts continuously as L_0^{-1} in the spaces with sub-exponential decay. Furthermore, it acts continuously*

$$L_b^{-1} \quad : \quad C(\mathbb{R} : H^s(\mathbb{R})) \longmapsto L^\infty_{loc}(\mathbb{R} : H^s(\mathbb{R}))$$

and for every $K > 0$, $s > 0$, one can find $C = C_K > 0$ such that

$$\|L_b^{-1} f\|_{L^\infty([-K,K]:H^s(\mathbb{R}))} \quad \leq \quad \frac{C}{\delta_0} \|f\|_{C([-K,K]:H^s(\mathbb{R}))},$$

for all $f \in C(\mathbb{R} : H^s(\mathbb{R}))$ and $\delta_0 > 0$.

Proof. We have the crucial step is based on the estimates near $t = \pm 1$:

$$\begin{aligned}
\|\hat{L_b^{-1}} \hat{f}(t, \cdot)\|_{L^2} \quad &\leq \quad C_0 \|f\|_{C([-K,K]:L^2(\mathbb{R}))} \sup_{\xi \in \mathbb{R}} \left(|1 + \mathrm{sign}(b)t|\right)^{\pm b(\xi)/2} \\
&\quad \times \left| \int_{-\mathrm{sign}(b)}^{t} \frac{1}{|1-s|^{1+\mathrm{sign}(b)/2} |1+s|^{1-\mathrm{sign}(b)/2}} \, ds \right| \\
&\leq \quad \frac{C}{|b|} \|f\|_{C([-K,K]:L^2(\mathbb{R}))}
\end{aligned}$$

where

$$C_0 = \sup_{\xi \in \mathbb{R}} \left(\left| \frac{1-t}{1+t} \right|^{b(\xi)/2} \int_{-\mathrm{sign}(b)}^{t} \frac{1}{(1-\tau)|1-\tau|^{b(\xi)/2} (1+s)|1+s|^{-b(\xi)/2}} \, ds \right) \leq \frac{2}{\delta_0}$$

□

5.5 Final Remarks

First we observe that our results remain valid for vector fields of the type

$$L = p(t)\partial_t + q(t, x)\partial_x$$

provided $q(t, x)$ is bounded for x, when $x \to \infty$. The approach follows the same ideas, but the arguments of the proofs become more involved in view of the use of theorems on global behaviour of solutions of ODEs. If q is not bounded, for $x \to \infty$, we have more restrictive conditions on the growth of the rhs f. For example, if $q(t, x)$ grows linearly in x (like SG first order hyperbolic pseudodifferential operators (see [52]), we have to require that the rhs $f(t, x)$ grows less than every $|x|^\gamma$, for every $\gamma > 0$. Next, we point out that if the rhs f decays to zero for $x \to \infty$, the right inverses L_j.

Next, as to possible multidimensional generalizations of the vector fields studied in the present work, we are also able to propose similar results for some classes of vector fields having smooth symmetries. E.g. consider the regular plane vector field $L = (t^2 - 15)(t^2 + 15)\partial_x - (t^2 - 25)(t^2 - 9)t\partial_t$. One can easily check that the rotations of L around the x axis in $\mathbb{R}^3$ with coordinates (t, x, y) gives rise to a regular vector field M having as separatrices the two cylinders $y^2 + t^2 = 9$ and $y^2 + t^2 = 25$. The cohomological equation $Mu = v$ hence is not solvable for every smooth function $v \in C^\infty(\mathbb{R}^3)$ because of the theorem of Duistermaat and Hormander but our techniques can be used to find weak solutions.

Finally, we point out to a natural problem related to the reduction of a perturbation $L + b(t, x, D)$ to L by means of global conjugation formally $J(t) \circ (L + b) \circ J^{-1}(t) = L$, with J being a global PDO or Fourier integral operator in $x \in \mathbb{R}^n$ depending smoothly on $t \in R \setminus I_L$, with singularities near $t = t_j$, S_j or S_{j+1} being separatrix strips. The example in Section 5.4 suggests that one should aim on estimates of $J(t)$ in $L^1_{loc}(\mathbb{R} : B(\mathbb{R}^n))$, where $B(\mathbb{R}^n)$ stands for some weighted Sobolev type space (see [56], [57], [58] and the references therein for global estimates in $\mathbb{R}^n$ for Fourier integral operators).

A

Transversality of linear PDOs

Linear homogeneous C^k PDOs $\mathcal{L}_\xi : C^1(M) \to C^0(M)$ are clearly in 1-1 correspondence with vector fields $\xi \in \mathfrak{X}^k(M)$. It is natural therefore to define their transversality to a hypersurface N as the transversality to N of the corresponding vector field, namely $\mathcal{L}_\xi$ is transversal to N at $n \in N$ if

$$\operatorname{span}\{\xi_n\} + T_n N = T_n M\,.$$

Let $f \in C^\infty(M)$ be any function regular at n such that N is the zero set of f in some neighbourhood of n. Then the condition $\operatorname{span}\{\xi_n\} + T_n N = T_n M$ translates in the fact that

$$\mathcal{L}_\xi f(n) \neq 0\,.$$

In case of general linear PDOs $\mathcal{L}_r : \Gamma^r F \to \Gamma^0 G$ of higher order, one can extend the definition thanks to the following observation: if $\nu \in C^\infty(E)$ is such that $\nu(e_0) = 0$, then

$$\mathcal{L}_r(\nu^r f)\Big|_{e_0,f_0} = L_{r,\nu} f\Big|_{e_0,f_0}$$

where $L_{r,\nu} : \Gamma^0 F \to \Gamma^0 G$ is the following linear zero-order operator:

$$(L_{r,\nu})^a_i = r!\Lambda_i^{a\alpha_1\ldots\alpha_r}\partial_{\alpha_1}\nu\cdots\partial_{\alpha_r}\nu\,.$$

Indeed all terms of $\mathcal{L}_r$ of order lower than r applied to $\nu^r f$ will leave at least one ν term which will vanish when evaluated at e_0, so the only surviving terms come from order r and only from those which act entirely on ν^r. We call $L_{r,\nu}$ the principal part of $\mathcal{L}_r$ with respect to ν. This justifies the following definitions:

Definition A.0.1 (Gromov, 1986)**.** *Given a 1-form $\lambda = (\lambda_1, \ldots, \lambda_m) \in T_{e_0}E$, we call the linear zero-order operator*

$$(L_{r,\lambda})^a_i = r!\Lambda_i^{a\alpha_1\ldots\alpha_r}\lambda_{\alpha_1}\cdots\lambda_{\alpha_r}$$

the principal part *of $\mathcal{L}_r$ at e_0 with respect to λ. We say that $\mathcal{L}_r$ is* transversal to the hyperplane $\ker\lambda \subset T_{e_0}E$ *if its principal part at e_0 with respect to λ is surjective. Given a higher-codimension plane $\pi = \cap_{i=1}^{l}\ker\lambda_i \subset T_{e_0}E$, we say that $\mathcal{L}_r$ is transversal to π if it is transversal to every hyperplane of $T_{e_0}E$ containing π. If $\mathcal{L}_r$ is not transversal to a plane π then it is said* tangential *to it.*

If $N \subset E$ is a submanifold of E, we say that $\mathcal{L}_r$ is transversal to N at e_0 *if it is transversal to $T_{e_0}N$. Finally, we say that N is* characteristic *for $\mathcal{L}_r$ if $\mathcal{L}_r$ is tangential to N at every point.*

Remark A.0.2. *From what said above it follows that $\mathcal{L}_r$ is transversal to a hypersurface N at e_0 iff its principal part with respect to ν at e_0, i.e. $L_{r,d\nu(e_0)}$, is surjective.*

Note that this definition agrees with the one given above for linear homogeneous first-order PDOs.

Example A.0.3. *Linear first-order PDOs L_ξ have always a characteristic manifold of dimension 1 given by the integral trajectories of the corresponding vector field ξ.*

Example A.0.4. *Consider the case of the Laplacian $\mathcal{L}_2 = \Delta_g = g^{\alpha\beta}\partial_\alpha\partial_\beta$ on a (pseudo)-Riemannian manifold M. The principal part of Δ_g with respect to a function $\nu \in C^\infty(M)$ is*

$$D_{g,\nu} = g^{\alpha\beta}\partial_\alpha\nu\partial_\beta\nu$$

Hence if g is Riemannian and ν_0 is a regular value for ν, Δ_g is transversal to $\nu^{-1}(\nu_0)$ at every point and moreover Δ_g has no characteristic hypersurfaces since $D_{g,\nu} = 0$ on every point of some hypersurface N would imply that ν is constant in some tubular nbhd of N. On the contrary, Δ_g can have characteristic hypersurfaces if M is pseudo-Riemannian: e.g. if $M = \mathbb{R}^2$ and $g = (dx)^2-(dy)^2$ then the "light-cones", i.e. the straight lines $d(x-y) = 0$ and $d(x+y) = 0$, are characteristics for Δ_g.

The following two lemmata are crucial for the proof of Theorem 2.4.17.

Lemma A.0.5 (Gromov, 1986)**.** *Let $\nu \in C^\infty(E)$ be regular at $e_0 \in E$, let $N = F^{-1}(F(e_0))$, so that N is a regular hypersurface close to e_0, and suppose that $\mathcal{L}_r : \Gamma^r F \to \Gamma^0 G$ is transversal to N at e_0. Then for every $k \in \mathbb{N}$ there exist a $s \in \mathbb{N}$ and two operators $A_k : \Gamma^{r+s}G \to \Gamma^r F$ and $B_k : \Gamma^s G \to \Gamma^0 G$, whose coefficients are rational functions of ν and its derivative up to order s and are regular at e_0, such that*

$$\mathcal{L}_r A_k + \nu^k B_k = i^0_{r+s}(G)\,.$$

Moreover this identity holds for small perturbations of ν and $\mathcal{L}_r$.

Proof. We prove the statement by induction. Clearly the theorem holds for $k=0$ with $s=0$ by putting $A_0=0$ and $B_0=i^0_0(G)$.

Now assume that there exist A_k and B_k of order s' such that

$$\mathcal{L}_r A_k + \nu^k B_k = i^0_{r+s'}(G)$$

and observe that

$$\mathcal{L}_r(\nu^{r+k} f) = \nu^k L_{rk,d\nu(e_0)}(f) + \nu^{k+1} R_{k,\nu}(f)\,,$$

where $L_{rk,d\nu(e_0)} = (r+k)!/r!\, L_{r,d\nu(e_0)}$ is invertible by hypothesis and $R_{k,\nu}$ is some linear PDO of order r.

Now define the PDOs of order $s=r+s'$ as

$$A_{k+1} = A_k + \nu^{r+k} L^{-1}_{rk,d\nu(e_0)} B_k\,,\quad B_{k+1} = -R_{k,\nu} L^{-1}_{rk,d\nu(e_0)} B_k\,.$$

Then

$$\begin{aligned}
&(\mathcal{L}_r A_{k+1} + \nu^{k+1} B_{k+1})g \\
&= \mathcal{L}_r(A_k + \nu^{r+k} L^{-1}_{rk,d\nu(e_0)} B_k)g + \nu^k(-R_{k,\nu} L^{-1}_{rk,d\nu(e_0)} B_k)g \\
&= g - \nu^k B_k g + \mathcal{L}_r(\nu^{r+k} L^{-1}_{rk,d\nu(e_0)} B_k g) - \nu^k R_{k,\nu} L^{-1}_{rk,d\nu(e_0)} B_k g \\
&= g - \nu^k B_k g + \nu^k B_k g + \nu^{k+1} R_{k,\nu} L^{-1}_{rk,d\nu(e_0)} B_k g - \nu^k R_{k,\nu} L^{-1}_{rk,d\nu(e_0)} B_k g \\
&= g\,,
\end{aligned}$$

namely

$$\mathcal{L}_r A_{k+1} + \nu^{k+1} B_{k+1} = i^0_{r+s}(G)\,.$$

$\square$

Lemma A.0.6 (Gromov, 1986). *Let* $F \xrightarrow{\pi_F} E$ *and* $G \xrightarrow{\pi_F} E$ *be vector bundles with* $\dim E = m$*,* $\dim F = m+q$ *and* $\dim G = m+q'$*. If* $q > q'$*, a generic linear PDO*

$$\mathcal{L}_r : \Gamma^r F \to \Gamma^0 G$$

has no characteristic submanifolds of positive codimension.

Proof. We must prove that the number of (closed) scalar conditions that a section $\Lambda_r : E \to \mathrm{Hom}(J^r F, G)$ must satisfy so that the corresponding PDO $\mathcal{L}_r$ has a characteristic manifold of positive codimension is larger than m.

Let $e_0 \in E$ be any point and consider the set $\mathcal{N}_k$ of all codimension-k submanifolds N passing through e_0 having, in coordinates, the form

$$x^{\mathrm{a}} = \nu^{\mathrm{a}}(x^{k+1},\dots,x^m)\,,\ \mathrm{a} = 1,\dots,k\,,$$

for some smooth map $\nu : \mathbb{R}^k \to \mathbb{R}^{m-k}$. If $\mathcal{L}_r$ is tangential to N at e_0 then the k matrices

$$\Lambda^{\mathrm{a}} = \Lambda_i^{a\alpha_1\dots\alpha_r} \partial_{\alpha_1}\nu^{\mathrm{a}} \dots \partial_{\alpha_r}\nu^{\mathrm{a}}$$

must all be of non-maximal rank, i.e. $\mathrm{rank}\,\Lambda^{\mathrm{a}} < q'$.

Each of these open conditions corresponds to $q-q'+1$ equations, for a total of $c(k)=k(q-q'+1)$ scalar closed conditions. Moreover there are $d(k)=k(m-k)$ linearly independent elements in the 1-jet of ν, so that the total dimension of the space of sections Λ_r being tangential to all codimension-k submanifolds of the type we are considering is

$$\left[\dim \operatorname{Hom}(J^k F, G) - k(q-q'+1)\right] + k(m-k)$$

and, correspondingly, its codimension is $k(q-q'+1)-k(m-k)$. Unfortunately this estimate proves our claim only for $q-q' > m-k-1-n/k$, which can be as large as we please for m large and m close to $m/2$.

In order to sharpen our estimate we use the fact that, for every $s \in \mathbb{N}$, the derivatives of order s of the $c(k)$ conditions above with respect to the $(m-k)$ coordinates provide extra conditions to be satisfied identically by the s-jets of all Λ_r which are tangential at e_0 to all submanifolds in $\mathcal{N}_k$. The number of conditions coming out from each relation $\operatorname{rank}\Lambda^{\mathrm{a}} < q'$ is now

$$c(k,s) = k(q-q'+1)\binom{m-k+s}{s}$$

while the dimension of the space of $(s+1)$-jets[1] of ν (minus the 0-jet, which does not appear anywhere) is

$$d(k,s) = k\binom{m-k+s+1}{s+1} - k\,.$$

Hence, if Λ_r is tangential to all submanifolds in $\mathcal{N}$ at e_0, its s-jet must satisfy a number of conditions equal to

$$c(k,s)-d(k,s) = k\binom{m-k+s}{s}\left[q-q'-\frac{m-k}{s+1}+\frac{1}{\binom{m-k+s}{s}}\right]$$

and clearly, since $q > q'$, this number can be easily made bigger than m for s big enough. $\square$

[1]Recall that within Λ^{a} appear the first derivatives of ν, so that in the s-jet of the relations $\operatorname{rank}\Lambda^{\mathrm{a}} < q'$ will appear the derivatives of ν up to the $s+1$-th order.

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